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Some Explicit Solutions of the Shape Equation

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Abstract. The consideration of the variational problem for minimization of the bending energy of membrane surfaces under some natural geometric constraints leads to the so called general shape equation. The available analytical solutions of this equation including a new one are reviewed in some detail.

Keywords: Classical differential geometry, plane curves, curvature, elasticity theory, membrane.
PACS: 02.40-k, 02.40.Hw, 46.25-y

INTRODUCTION

The description of the equilibrium shapes and flow-induced deformations of vesicles and biological cells membranes is a long-standing problem in biophysics, cell biomechanics and bioengineering. Especially, the biconcave, disk-like resting shapes of healthy blood cells attracted a lot of attention. Many authors have computed these shapes of elastic axisymmetric membranes consisting of a lipid bilayer by minimization of an energy functional defined in terms of membrane curvatures. E.g., already in 1970, Canham had proposed in [3] that the "curvature elasticity" is the main shape-controlling factor.

His model is based on the analogy with the simple beam in the plane (Euler elastica) which equilibrium equation can be derived by considering the variational problem of minimizing its bending energy $\mathscr{E}$ given by the integral of the squared curvature κ

$$\mathscr{E} = \int \kappa^2(s)\,\mathrm{d}s \tag{1}$$

taken over the length of the beam. Assuming that the membrane exhibits pure bending in two perpendicular planes, one can obtain the stored elastic energy as an integral of the sum of the squares of the two principal curvatures

$$\mathscr{E} = \int (\kappa_1^2 + \kappa_2^2)\,\mathrm{d}A \tag{2}$$

taken over the membrane surface. In 1973 Helfrich [6] extended Canham's model by suggesting the so-called spontaneous curvature model according to which the equilibrium shapes of a lipid vesicle are determined by the extremals of the curvature (shape)

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On the Plane Curves whose Curvature Depends on the Distance from the Origin

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Abstract. Here we suggest and have exemplified a simple scheme for reconstruction of a plane curve if its curvature belongs to the class specified in the title by deriving explicit parametrization of Bernoulli's lemniscate and newly introduced co-lemniscate curve in terms of the Jacobian elliptic functions. The relation between them and with the Bernoulli elastica are clarified.

Keywords: classical differential geometry, plane curves, curvature, elasticity theory
PACS: 02.40-k, 02.40.Hw, 46.25-y

INTRODUCTION

The most fundamental existence and uniqueness theorem in the theory of plane curves states that a curve is uniquely determined (up to Euclidean motion) by its curvature given as a function of its arc-length (see [1, p. 296] or [8, p. 37]). The simplicity of the situation however is quite elusive because in many cases it is impossible to find the sought-after curve explicitly. Having this in mind, it is clear that the situation is even more complicated if the curvature is given as a function of its position. Viewing the Frenet-Serret equations as a fictitious dynamical system it was proven in [10] that when the curvature is given just as a function of the distance from the origin the problem can always be reduced to quadratures. The cited result should not be considered as entirely new because Singer [9] had already shown that in some cases it is possible that such curvature gets an interpretation of a central potential in the plane and therefore the trajectories could be found by the standard procedures in classical mechanics. The approach which we will follow here, however is entirely different from the group-theoretical [10] or mechanical one [9] proposed in the above cited papers. The method is illustrated on the most natural examples in the class of curves whose curvatures depend solely on the distance from the origin. Here we consider the case in which the function in question is

$$\kappa = \sigma r, \qquad r = |\mathbf{x}| = \sqrt{x^2 + z^2} \tag{1}$$

where x, z are the Cartesian coordinates in the plane XOZ which have to be considered as functions of the arc-length parameter s, and σ is assumed to be a positive real constant.

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MATHEMATIQUES
Géometrie différentielle

ON BERNOULLI'S LEMNISCATE AND CO-LEMNISCATE

Mariana Hadzhilazova, Ivaïlo M. Mladenov

(*Submitted by Academician P. Popivanov on February 23, 2010*)

Abstract

Explicit parameterizations of Bernoulli's Lemniscate and newly introduced Co-Lemniscate curve in terms of the Jacobian elliptic functions are derived. The relation between them and with the Bernoulli elastica are clarified.

Key words: plane curves, Bernoulli's Lemniscate, Co-Lemniscate, elastica
2010 Mathematics Subject Classification: 14H50, 53A04, 74K10

1. Introduction. The fundamental existence and uniqueness theorem in the theory of plane curves states that a curve is uniquely determined (up to Euclidean motion) by its curvature given as a function of its arc-length (see [1], p. 296 or [6], p. 37). The simplicity of the situation, however, is elusive as in many cases it is impossible to find the curve explicitly. Having this in mind, it is clear that if the curvature is given as a function of its position, the situation is even more complicated. Viewing the Frenet-Serret equations as a ficticious dynamical system, it was proven in [8] that, when the curvature is given just as a function of the distance from the origin the problem can always be reduced to quadratures. This last result is not entirely new as Singer [7] had already shown that in some cases it is possible for such curvature to get an interpretation of a central potential in the plane and, therefore the trajectories (the sought-after curves) could be found by the standard procedures in classical mechanics. Here the approach which we will follow, however, is entirely different from the group-theoretical [8] or mechanical one [7] proposed in the above cited papers. The

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Доклади на Българската академия на науките
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MATHEMATIQUES

Géométrie différentielle

ON THE GENERALIZED STURMIAN SPIRALS

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(*Submitted by Academician P. Popivanov on November 30, 2010*)

Abstract

The Sturm spirals which can be introduced as those plane curves whose curvature radius is equal to the distance from the origin are embedded into one-parameter family of curves. Explicit parametrization of the ordinary Sturmian spirals along with that of a wider family of curves are found and depicted graphically.

Key words: Sturm spirals, plane curves, equiangular spiral, logarithmic spiral, Norwich spiral

2010 Mathematics Subject Classification: 14H50, 53A04, 74K10

1. Introduction. The fundamental existence and uniqueness theorem in the theory of plane curves states that a curve is uniquely determined (up to Euclidean motion) by its curvature given as a function of its arc-length (see [1], p. 296 or [8], p. 37). The simplicity of the situation, however, is elusive as in many cases it is impossible to find the curve explicitly. Having that in mind, it is clear that if the curvature is given as a function of its position, the situation is even more complicated. A nice exception is provided by the Euler's elastica curves [3,6,7] whose curvature actually is a function of the distance from a fixed line in the Euclidean plane. Viewing the Frenet-Serret equations as a fictitious dynamical system, it was proven in [11] that when the curvature is given just as a function of the distance from the origin the problem can always

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On Some Deformations of the Cassinian Oval

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Abstract. The work is concerned with the determination of explicit parametric equations of several plane curves whose curvature depends solely on the distance from the origin. Here we suggest and exemplify a simple scheme for reconstruction of a plane curve if its curvature belongs to the above-mentioned class. Explicit parameterizations of generalized Cassinian ovals including also the trajectories of a charged particle in the field of a magnetic dipole are derived in terms of Jacobian elliptic functions and elliptic integrals.

Keywords: classical differential geometry, plane curves, curvature, Cassinian oval, magnetic dipole
PACS: 02.40-k, 02.40.Hw, 46.25-y

INTRODUCTION

Remarkably, the curvature of a lot of the famous plane curves (see [5, 14]), such as conic sections, Bernoulli's lemniscate, Cassinian ovals and many others, depends solely on the distance from a certain point in the Euclidean plane, which may be chosen as its origin.

The most fundamental existence and uniqueness theorem in the theory of plane curves states that a curve is uniquely determined (up to Euclidean motion) by its curvature given as a function of its arc-length (see [3, p. 296] or [9, p. 37]). The simplicity of the situation however is quite elusive because in many cases it is impossible to find the sought-after curve explicitly. Having this in mind, it is clear that if the curvature is given by a function of its position the situation is even more complicated. Viewing the Frenet-Serret equations as a ficticious dynamical system it was proven in [11] that when the curvature is given just as a function of the distance from the origin the problem can always be reduced to quadratures. The cited result should not be considered as entirely new because Singer [10] had already shown that in some cases it is possible that such curvature gets an interpretation of a central potential in the plane and therefore the trajectories could be found by the standard procedures in classical mechanics. The approach which we will follow here, however is entirely different from the group-theoretical [11] or mechanical [10] ones proposed in the aforementioned papers. The method is illustrated on a class of curves whose curvature depend solely on the distance from the origin.

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Analytic Description of the Equilibrium Shapes of Elastic Rings Under Uniform Hydrostatic Pressure

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Abstract. The parametric equations of the plane curves determining the equilibrium shapes that a uniform inextensible elastic ring could take subject to a uniform hydrostatic pressure are presented in an explicit analytic form. The determination of the equilibrium shape of such a structure corresponding to a given pressure is reduced to the solution of two transcendental equations. The shapes with points of contact and the corresponding (contact) pressures are determined by the solutions of three transcendental equations. The analytical results presented here confirm many of the previous numerical results on this subject but the results concerning the shapes with lines of contact reported up to now are revised.

Keywords: Elastic ring, hydrostatic pressure, equilibrium shapes, parametric equations
PACS: 02.30.Hq, 02.40.Hw, 46.32.+x, 46.70.Hg

INTRODUCTION

The present paper addresses the problem for determination of the equilibrium shapes of a circular inextensible elastic ring subject to a uniformly distributed external force that acts normally to the ring in the ring plane.

Maurice Lévy [22] was the first who stated and studied the problem under consideration and reduced the determination of the foregoing equilibrium shapes in polar coordinates to two elliptic integrals for the arclength and polar angle regarded as functions of the squared radial coordinate. He found also several remarkable properties of the equilibrium ring shapes and concluded that if the pressure p is such that $p < (9/4)(D/\rho^3)$, where D and ρ are the ring bending rigidity and radius of the undeformed shape, respectively, then the ring possesses only the circular equilibrium shape.

Later on, Halphen [17] and Greenhill [15] derived exact solutions to this problem in terms of the Weierstrass elliptic functions on the ground of complicated analyses of the properties of the aforementioned elliptic integrals. Halphen (see [17, p. 235]) found out that non-circular shapes with $n \geq 2$ axes of symmetry are possible only for pressures greater than $p_n = (n^2 - 1)(D/\rho^3)$. Halphen [17] and Greenhill [15] presented also several examples of non-circular equilibrium ring shapes. It should be noted, however, that the exact solutions reported in [17, 15], representing the polar angle as a function of the radius, appeared to be intractable and many researchers continued searching exact solutions [4–10], while others used various approximations [24, 13] on the way

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Cell Membranes Under Hydrostatic Pressure Subjected to Micro-Injection

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Abstract. The work is concerned with the determination of the mechanical behaviour of cell membranes under uniform hydrostatic pressure subject to micro-injections. For that purpose, assuming that the shape of the deformed cell membrane is axisymmetric a variational statement of the problem is developed on the ground of the so-called spontaneous curvature model. In this setting, the cell membrane is regarded as an axisymmetric surface in the three-dimensional Euclidean space providing a stationary value of the shape energy functional under the constraint of fixed total area and fixed enclosed volume. The corresponding Euler-Lagrange equations and natural boundary conditions are derived, analyzed and used to express the forces and moments in the membrane. Several examples of such surfaces representing possible shapes of cell membranes under pressure subjected to micro injection are determined numerically.

Keywords: Cell membrane, micro-injection, spontaneous-curvature model, axisymmetric shapes, forces and moments, bending energy, variational statement, Euler-Lagrange equations, natural boundary conditions, jump conditions
PACS: 87.16.D-, 02.40.Hw, 02.30.Hq, 02.30.Ik

INTRODUCTION

The 2010 Nobel Prize in Physiology or Medicine was awarded to Robert Edwards for the development of human in-vitro fertilization in 1977 (Louise Brown, the world's first "test tube baby", was born on 25 July, 1978). On the other side, genetic engineering is a rapidly developing area of biology in the past 30 years aimed in creation of transgenic organisms with desired properties. Recently, the controlled delivery of diamond and gold nanoparticles within a single cell has being developed (see, e.g. [7]), and is expected to become a broadly applicable tool for therapy, since these nanoparticles being not toxic can be used as carriers for therapeutics, proteins, antibodies, DNA and other biological agents. Presently, these three fields of the human activity involve the intracellular delivery of substances by micro-injection. During the process of a micro-injection, a micro pipette pierces the cell membrane and releases substances within the cell interior. The success of a micro-injection depends mainly on the mechanical properties of the injected cell membrane and on the specific way of interaction between the membrane and the holding and injection pipettes.

Observing the literature on micro-injections of cells one realizes that large cells are the most often studied, typical examples being the zebrafish and mouse embryos. The

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Analytic description and explicit parametrisation of the equilibrium shapes of elastic rings and tubes under uniform hydrostatic pressure

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ABSTRACT

The parametric equations of the plane curves determining the equilibrium shapes that a uniform inextensible elastic ring or tube could take subject to a uniform hydrostatic pressure are presented in an explicit analytic form. The determination of the equilibrium shape of such a structure corresponding to a given pressure is reduced to the solution of two transcendental equations. The shapes with points of contact and the corresponding (contact) pressures are determined by the solutions of three transcendental equations. The analytic results presented here confirm many of the previous numerical results on this subject but the results concerning the shapes with lines of contact reported up to now are revised.

1. Introduction

In the present paper, the problem for determination of the equilibrium shapes of a circular inextensible elastic ring subject to a uniformly distributed external force that acts normally to the ring in the ring plane is addressed. This problem is also referred to as the stability problem or buckling of the circular shape of the ring and the other equilibrium shapes are called buckled [1–3]. It is also known (see, e.g., [4–6]) that if a cylindrical elastic shell of circular cross section (i.e., a tube) is subject to a uniform external pressure, which is normal to its middle surface, then the typical cross section of the deformed tube takes the same shapes as the axis of a deformed elastic ring does provided that the latter is a simple curve (i.e., a curve without intersections). Therefore, here the term "ring" will be used to indicate both a proper ring and a tube. It should be noted also that in the majority of the works in this field, the distributed force acting on a ring is called pressure as in the case of a shell. Following this tradition, we will use the same term in the present study remembering that pressure means force per unit length in the case of a ring and force per unit area in the case of a shell.

Maurice Lévy [7] was the first who stated and studied the problem under consideration and reduced the determination of the foregoing equilibrium shapes in polar coordinates to two elliptic integrals for the arclength and polar angle regarded as functions of the squared radial coordinate. He found also several remarkable properties of the equilibrium ring shapes and obtained that if the pressure p is such that $p<(9/4)(D/\rho^3)$, where D and ρ are the ring bending rigidity and radius of the undeformed shape, respectively, then the ring possesses only the circular equilibrium shape.

Later on, Halphen [8] and Greenhill [9] derived exact solutions to this problem in terms of Weierstrass elliptic functions on the ground of complicated analyses of the properties of the aforementioned elliptic integrals. Halphen (see [8, p. 235]) found out that non-circular shapes with $n\geq 2$ axes of symmetry are possible only for pressures greater than $p_n=(n^2-1)(D/\rho^3)$. Halphen [8] and Greenhill [9] presented also several examples of non-circular equilibrium ring shapes. It should be noted, however, that the exact solutions reported in [8,9], representing the polar angle as a function of the radius, appeared to be intractable and many researchers continued searching exact solutions [1,10–15], while others used various approximations [2,4,6,16] on the way to determine the equilibrium shapes of the ring.

Carrier [1] was the first who reconsidered the foregoing problem for the buckling of an elastic ring about half a century after the works by Lévy, Halphen and Greenhill. He expressed the curvature of the deformed ring in terms of Jacobi cosine function [17] involving several unknown parameters to be determined by a system of algebraic equations. However, he succeeded to find approximate solutions to this system only for small deflections from the undeformed circular ring shape (see the exhaustive analysis provided recently by Adams [11] who has criticised and developed Carrier's work [1]).

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Traveling Wave Solutions of the Gardner Equation and Motion of Plane Curves Governed by the mKdV Flow

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Abstract. The Gardner equation is well-known in the mathematical literature since the late sixties of 20th century. Initially, it appeared in the context of the construction of local conservation laws admitted by the KdV equation. Later on, the Gardner equation was generalized and found to be applicable in various branches of physics (solid-state and plasma physics, fluid dynamics and quantum field theory). In this paper, we examine the travelling wave solutions of the Gardner equation and derive the full set of solutions to the corresponding reduced equation in terms of Weierstrass and Jacobi elliptic functions. Then, we use the travelling wave solutions of the focusing mKdV equation and obtain in explicit analytic form exact solutions of a special type of plane curve flow, known as the mKdV flow.

Keywords: Gardner equation, KdV equation, modified KdV equation, travelling wave solutions, Weierstrass and Jacobi elliptic functions, Motion of plane curves, mKdV flow
PACS: 02.30.Jr, 02.30.Hq, 02.30.Ik

INTRODUCTION

The nonlinear evolution partial differential equation

$$u_t + uu_x + \frac{1}{6}\varepsilon^2 u^2 u_x + u_{xxx} = 0, \qquad \varepsilon \in \mathbb{R} \tag{1}$$

usually referred to as the Gardner equation (see [1]), was introduced almost half a century ago in the fist one [2] of a series of works (see also [3]) by Miura, Gardner, Kruskal and coauthors devoted to the study of properties and solutions of the celebrated Korteweg-de Vries (KdV) equation [4]

$$u_t + uu_x + u_{xxx} = 0 \tag{2}$$

and its simplest modification

$$u_t + u^2 u_x + u_{xxx} = 0 \tag{3}$$

currently known as the (focusing) modified Korteweg-de Vries (mKdV) equation. These equations have a great deal in common with the Camassa-Holm equation, but there are significant differences as well [5]. Here and in what follows, the subscripts denote partial differentiations of the dependent variable (unknown function) $u = u(x,t)$ with respect to the indicated independent variables x and t.

In the present paper, by "Gardner equation" we assume a combination of the aforementioned three equations of the form

$$u_t + \alpha_1 u_{xxx} + \alpha_2 u_x + \alpha_3 uu_x + \alpha_4 u^2 u_x = 0, \qquad \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R}. \tag{4}$$

Thus, by setting $\alpha_1 = \alpha_3 = 1$, $\alpha_2 = 0$, $\alpha_4 = (1/6)\varepsilon^2$ in Eq. (4) one obtains the genuine Gardner equation (1), the setting $\alpha_1 = 1$, $\alpha_2 = \alpha_4 = 0$, $\alpha_3 = 1$ yields the KdV equation (2) and, finally, choosing $\alpha_1 = 1$, $\alpha_2 = \alpha_3 = 0$, $\alpha_4 = 1$ one gets to the mKdV equation (3). It should be noted that equations of form (4) have attracted a lot of attention in recent years being frequently called "extended KdV (mKdV) equations" (see, *e.g.*, [6, 7, 8]) or "combined KdV–mKdV equations", see, *e.g.*, [9, 10, 11, 12, 13].

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Deformation of injected vesicles adhering onto flat rigid substrates

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ARTICLE INFO

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ABSTRACT

This study is concerned with the determination of the mechanical behaviour of closed fluid lipid bilayer membranes (vesicles) under a uniform hydrostatic pressure, pressed against and adhering onto a flat homogeneous rigid substrate. Assuming that the initial and deformed shapes of the vesicle are axisymmetric, a variational statement of the problem is developed on the ground of the so-called spontaneous curvature model. In this setting, the vesicle is regarded as a closed surface in the three-dimensional Euclidean space and its equilibrium shapes are supposed to provide stationary values of the bending energy functional under the constraint of fixed total area. The corresponding Euler–Lagrange equations and natural boundary conditions are derived, the work done by the pressure being taken into account, and used to determine the forces and moments in the membrane. Several examples of surfaces representing possible equilibrium shapes of so loaded membranes are determined numerically.

1. Introduction

The success of cell manipulations depends mainly on the mechanical properties of the cell membrane and on the specific way of interaction between the membrane and the other devices. To this end, the determination of the mechanical behaviour of the cells is of primary interest. An apparent approach to this analysis is theoretical determination of certain cell equilibrium shapes and their comparison with experimental observations, a typical example being presented by Lu et al. [1].

The choice of a model to study the equilibrium shapes of cells depends on the time scale of the phenomena due to the existence of active processes of permeation of matter through the cell membrane [2]. Phenomena that are much faster than the active transfer processes are reasonable to treat using models of deformation that maintain fixed cell volume. In this case it is widely accepted that the deformation of the cell membrane is localized, typical examples being Lu et al. [1], Boulbitch [3], Bo [4], Sun et al. [5], Tan et al. [6] and Wan et al. [7]. On the other hand, if a phenomenon is much slower than the permeation through the membrane it is reasonable to consider a model that maintains fixed membrane area. In the present study, we consider cells pressed against a rigid wall and are interested in equilibrium shapes that are attained after the finishing of the transient transfer of matter through the cell membrane. For this purpose, we employ the simplest model of cells—closed fluid lipid bilayer membranes (vesicles).

A general theoretical model for deformation of lipid bilayer membranes was proposed by Helfrich [8]. This model, usually referred to as the spontaneous curvature model, is widely acknowledged and used by many authors to study stresses and strains in cell membranes (see, e.g., the exhaustive surveys [9–14]). The corresponding partial differential equations determining the equilibrium shapes of closed lipid bilayer membranes (vesicles) subjected to hydrostatic pressure is derived in 1989 by Ou-Yang and Helfrich [15]. Later on, Capovilla et al. [16] and Tu and Ou-Yang [17,18] have extended the foregoing

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A RELATION BETWEEN THE CYLINDRIC FLUID MEMBRANES AND THE MOTIONS OF PLANAR CURVES

PETKO I. MARINOV AND IVAÏLO M. MLADENOV

Presented by Ivaïlo M. Mladenov

Abstract. We observe a relation between the mKdV equation and the cylindrical equilibrium shapes of fluid membranes. In our setup mKdV arises from the study of the evolution of planar curves.

1. Introduction

The goal of this paper is to unify and extend the results presented in [5] and [8]. It also shows a connection between two problems that appear unrelated.

The first problem comes from the study of equilibrium shapes of fluid membranes. One starts with a functional proposed by Helfrich (see [2], [8]) and studies the corresponding Euler-Lagrange equation. The equilibrium shapes are given as the extremals of the functional

$$\mathcal{F} = \frac{k_c}{2}\int_S (2H + \mathbb{h})^2 \mathrm{d}A + k_G \int_S K \mathrm{d}A + \lambda \int_S \mathrm{d}A + p \int \mathrm{d}V. \tag{1}$$

Notice that $\mathcal{F}$ is closely related to the Willmore energy functional. The Euler-Lagrange equation associated with $\mathcal{F}$ is as follows

$$2k_c\Delta_S H + k_c(2H + \mathbb{h})(2H^2 - \mathbb{h}H - 2K) - 2\lambda H + p = 0. \tag{2}$$

Here H and K are the mean and Gauss curvatures respectively, k_c and k_G - bending and Gaussian constant rigidity of the membrane, $\mathbb{h}$ is spontaneous curvature constant, p and λ - Lagrange multipliers corresponding to fixed volume and total membrane area and Δ_S is the surface Laplacian on the interface of the membrane. The nature of this equation is complex as it involves the surface Laplacian of the mean curvature which makes it a fourth-order non-linear PDE. However, as always, the symmetry of the problem reduces the equation and in the special case of cylindrical membranes it becomes the ordinary differential equation

$$2\frac{\mathrm{d}^2\kappa}{\mathrm{d}s^2} + \kappa^3 - \mu\kappa - \sigma = 0. \tag{3}$$

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An approach for decomposition of finite rotations

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Based on Lie groups theory, this work considers the problem of decomposition of a given rotation into three successive finite rotations with prescribed in advance axes.

1 Introduction

The different parameterizations of the rotation group SO(3) influence strongly the efficiency of the kinematic and dynamic models as at one rigid body so in multibody mechanical systems. Various analytical representations of rotations are obtained when the rotations are expressed by defining their action on vectors, quaternions, spinors, etc [1]. On the other hand the most used explicit parameterizations of the rotations are via: Eulerian angles in the classical 3-1-3 sense and all other combinatons like: 3-2-3, 3-2-1, 1-2-3 and etc., Bryant angles, Eulerian parameters, Cayley-Klein parameters and etc. (see [2] - [4]). To find the resultant axis and angle of rotation after two, three or more finite partial rotations is a very important but trivial problem in multibody mechanics. The inverse problem however, namely, to decompose a finite rotation into three partial rotations about prescribed axes is a more difficult one and quite important in motion planning in the group of rotations and inverse kinematic problem at the manipulator systems as well. The present paper gives explicit formulae in solving this problem using vector-like parameterization of the rotation group [5] - [8].

2 Problem statement and algorithm realization

As an exception in the three-dimensional space, there exists a map (actually isomorphism) between vectors and skew-symmetric matrices, i.e., if $c \in \mathbb{R}^3$, we have $c \to c^{\times}$, where $c^{\times}$ is the corresponding skew-symmetric matrix. Then we may write the SO(3) matrix in the form

$$O = O(c) = (I + c^{\times})(I - c^{\times})^{-1} = \frac{(1 - c^2)I + 2\,c \otimes c + 2\,c^{\times}}{1 + c^2}\,. \tag{1}$$

Here I is the 3×3 identity matrix, $c \otimes c$ denotes the diadic matrix formed by the vector c. The formula above provides us with an explicit parameterization of SO(3). The vector c is called a **vector-parameter**. It is parallel to the axis of rotation and its module $\|c\|$ is equal to $\tan(\alpha/2)$, where α is the angle of rotation. The so defined vector-parameters form a Lie group with the following composition law

$$c' = \langle c_1, c_2 \rangle = \frac{c_1 + c_2 + c_1 \times c_2}{1 - c_1\,c_2}\,. \tag{2}$$

The symbol "$\times$" means cross product of vectors. Conjugating with elements from the $\mathsf{SO}(3)$ group leads to linear transformations in the vector-parameter space $O(c)\,O(c')\,O^{-1}(c) = O(c)\,O(c')\,O^{T}(c) = O(c'')$ where $c'' = O(c)\,c' = O_c\,c'$ and "T" symbolizes the transposed matrix. Such a parameterization in the Lie group theory is called a natural one. It is worth mentioning also that no other parameterization possesses neither this property nor a manageable superposition law.
Problem statement: Given a vector-parameter c and three axes specified by their unit vectors, find the respective rotation angles through which the initial vector has to be decomposed.

Let us denote the three unknown vectors by $c_1 = u\,\hat{c}_1$, $c_2 = v\,\hat{c}_2$ and $c_3 = w\,\hat{c}_3$ where $\hat{c}_i$, $i = 1, 2, 3$ are the unit vectors along the prescribed axes of rotations. Having in mind the defining relation $O(c) = O(c_1)O(c_2)O(c_3)$ the following relations for the dot products of vectors are valid (further on by (p, q) we will denote the dot product of the vectors p and q)

$$(\hat{c}_1, O(c)\hat{c}_3) = (\hat{c}_1, O(c_2)\hat{c}_3) \tag{3}$$

$$(\hat{c}_2, O(c)\hat{c}_3) = (\hat{c}_2, O(c_1)O(c_2)\hat{c}_3) = (O^T(c_1)\hat{c}_2, O(c_2)\hat{c}_3) \tag{4}$$

$$(\hat{c}_1, O(c)\hat{c}_2) = (\hat{c}_1, O(c_2)O(c_3)\hat{c}_2) = (O^T(c_2)\hat{c}_1, O(c_3)\hat{c}_2) \tag{5}$$

$$(\hat{c}_3, O(c)\hat{c}_3) = (\hat{c}_3, O(c_1)O(c_2)\hat{c}_3) = (O^T(c_1)\hat{c}_3, O(c_2)\hat{c}_3). \tag{6}$$

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МОДЕЛ НА ХЕЛФРИХ ЗА ФОРМАТА НА БИОЛОГИЧНИТЕ МЕМБРАНИ: ГРУПОВ АНАЛИЗ, ОПРЕДЕЛЯЩА СИСТЕМА И ДОПУСТИМИ СИМЕТРИИ *

Владимир Пулов, Мариана Хаджилазова,
Красимира Кърджилова, Валентин Люцканов, Ивайло Младенов

HELFRICH`S SHAPE MODEL OF BIOLOGICAL MEMBRANES: GROUP ANALYSIS, DETERMINING SYSTEM AND SYMMETRIES

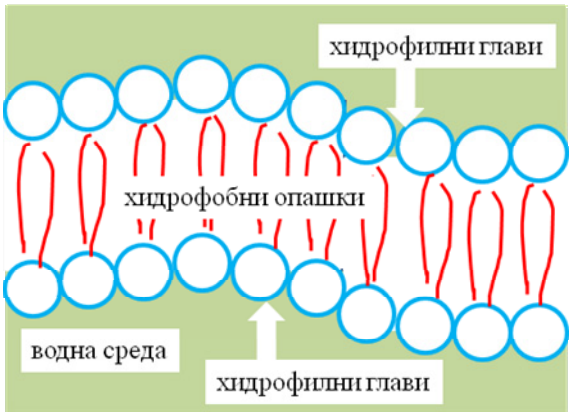
Abstract: Helfrich's membrane shape model [1, 2] is considered from group-theoretical viewpoint [5, 6]. Using conformal metric on the surface the Helfrich`s model is represented by a system of four nonlinear partial differential equations of second order of the derivatives [3, 4]. In order to construct the determining system for the symmetries of the conformal metric representation of the Helfrich's model we took advantage of the *MATHEMATICA* package *LieSymm-PDE* [7]. We obtained a determining system consisting of 271 equations. Using the computer algebra system *MATHEMATICA* we solved most of the equations in a semi-automatic way. As a result only 27 equations remained unsolved. We present here these equations postponing their solution for future work.
Keywords: биологични мембрани, групов анализ на диференциални уравнения, компютърна алгебра biological membranes, group analysis of differential equations, computer algebra

I. УРАВНЕНИЕ НА ХЕЛФРИХ ЗА ФОРМАТА НА БИОЛОГИЧНИТЕ МЕМБРАНИ

Уравнението на Хелфрих

$$\Delta H + 2(H^2 + \mathbb{h}H - K)(H - \mathbb{h}) - \frac{2\lambda H}{k} + \frac{p}{k} = 0 \qquad (1)$$

е основно уравнение в теорията на клетъчните биомембрани [1, 2]. С него се описват равновесните форми на най-простите затворени мембранни структури – така наречените везикули. В биологията везикула се нарича всяко малко мехурче с размери от 15 μm до 0.5 cm и дебелина на стената до 4-5 nm. Везикулите се формират във водна среда, най-често от молекули на фосфолипиди. Фосфолипидите са вещества изградени от амфифилни молекули. Амфифилни се наричат молекулите, които съчетават в себе си, както хидрофилни, така и хидрофобни свойства. При фосфолипидите това се дължи на добре обособените в двата края на молекулата хидрофилна глава и две хидрофобни опашки (Фигура 1).



Фигура 1. Двоен фосфолипиден слой.

Поставени във воден разтвор фосфолипидните молекули образуват двоен слой, при който хидрофилните им глави сочат навън към разтвора, а опашките им, за да нямат пряк контакт с водните молекули, са обърнати към вътрешността на слоя (Фигура 1). След като достигне определена критична площ, двойният фосфолипиден слой, който първоначално е плосък, започва спонтанно да се огъва, докато образува затворена кухина, изпълнена с течност – това е везикулата.

В уравнението на Хелфрих (1) двойният липиден слой на везикулите е представен като повърхнина S със средна кривина H и гаусова кривина K. Физичните параметри,

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Equilibrium shapes of fluid membranes and carbon nanostructures

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Abstract

The present chapter concerns the continuum modelling of the mechanical behaviour and equilibrium shapes of two types of nano-scale objects: fluid lipid bilayer membranes and carbon nano-structures. A unified continuum model is used to handle four different case studies. Two of them consist in representing in analytic form cylindrical and axisymmetric equilibrium configurations of single-wall carbon nanotubes and fluid lipid bilayer membranes subjected to uniform hydrostatic pressure. The third one is concerned with determination of possible shapes of junctions between a single-wall carbon nanotube and a flat graphene sheet or another single-wall carbon nanotube. The last one deals with the mechanical behaviour of closed fluid lipid bilayer membranes (vesicles) adhering onto a flat homogeneous rigid substrate subjected to micro-injection and uniform hydrostatic pressure.

Keywords: *graphene, carbon nanotubes and nanostructures, junctions, bending energy, natural boundary conditions, cell injection, adhesion, equilibrium shapes*

1. Introduction

This chapter is concerned with the mechanical behaviour and shape analysis of two types of nano-scale objects of quite different physical and chemical nature: fluid membranes (FM's) and carbon nanostructures (CNS's).

Here, by a fluid membrane we mean a membrane formed in aqueous solution by a bilayer of lipid molecules, which are in a fluid state, i.e. the molecules can move freely within the monolayer they belong to. The structure of the bilayer is such that the hydrophobic tails of the lipid molecules situated in different monolayers face one another to form a semi-permeable core, while their hydrophilic heads face the aqueous solutions on either side of the membrane. It is well-known that the lipid bilayer is the main structural component of all biological membranes, the closed lipid bilayer membranes

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Serret's Curves, their Generalization and Explicit Parametrizations

Ivaïlo M. Mladenov, Mariana Ts. Hadzhilazova,
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Abstract. Here we apply our original scheme for the reconstruction of plane curves whose curvatures are specified by functions of the radial coordinate to the curves introduced by J.-A. Serret. These curves are associated with the natural numbers and we extend their definition in order to include them into a family of curves depending on two continuous real parameters. The explicit parametrization of this new class of curves is presented as well.

Mathematics Subject Classification (2010). Primary 53A04; Secondary 53A55, 53A17.

Keywords. Classical differential geometry, plane curves, curvature, Frenet-Serret equations.

1. Introduction

Long time ago Serret [1] has described a family of plane algebraic curves in response to a question raised by Legendre. The problem was to find algebraic curves other than the lemniscate, such that their arc lengths are expressed by elliptic integrals of the first kind, and Serret claimed that he has found all such rational curves. Besides he provides a mechanical procedure [2] for their construction which will be described in the next Section. Before that we will mention that the original Serret curves were indexed by natural numbers but Liouville [3] had recognized immediately that rational numbers are suited as well as they also lead to algebraic curves. This has been further elucidated in Krohs' dissertation [4]. Here, we extend the definition of Serret's curves from discrete to continuous two-parameter family and present their explicit parametrizations.

Actually, the organization of the paper is as follows. The next section presents the mechanical construction of Serret's curves followed by another one in which the Frenet-Serret equations are formulated in Cartan moving frame. Then we outline

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VECTOR DECOMPOSITIONS OF ROTATIONS

DANAIL BREZOV, CLEMENTINA MLADENOVA AND IVAÏLO MLADENOV

Presented by Ivaïlo M. Mladenov

Abstract. Here we derive analytic expressions for the scalar parameters which appear in the generalized *Euler* decomposition of the rotational matrices in $\mathbb{R}^3$. The axes of rotations in the decomposition are almost arbitrary and they need only to obey a simple condition to guarantee that the problem is well posed. A special attention is given to the case when the rotation is decomposable using only two rotations and for this case quite elegant expressions for the parameters were derived.

In certain cases one encounters infinite parameters due to the rotations by an angle π (the so called *half turns*). We utilize both geometric and algebraic methods to obtain those conditions that can be used to predict and deal with various configurations of that kind and then, applying *l'Hôpital's* rule, we easily obtain the solutions in terms of linear fractional functions. The results are summarized in two Tables and a flowchart presenting in full details the procedure.

Contents

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Geometry of the anisotropic minimal surfaces

Ivaïlo M. Mladenov and Mariana Ts. Hadzhilazova

Abstract

A simple modification of the surface tension in the axisymmetric case leads to analogues of the Delaunay surfaces. Here we have derived an explicit parameterization of the most simple case of this new class of surfaces which can be considered as a generalization of the catenoids. The geometry of these surfaces depends on two real parameters and has been studied in some detail.

1 Introduction

In aqueous solution, amphiphilic molecules (e.g., phospholipids) form bilayers, the hydrophilic heads of these molecules being located in both outer sides of the bilayer, which are in contact with the liquid, while their hydrophobic tails remain at the interior. The handbook [7] is a good starting point for learning more about this phenomena.

A bilayer may form a closed membrane which bear the name vesicle. Vesicles constitute a well-defined and sufficiently simple model system for studying the basic physical properties of the more complex cell biomembranes which

Key Words: Delaunay surfaces, membranes, general shape equation.
2010 Mathematics Subject Classification: 53A05, 53A10, 53B50.
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MATHEMATIQUES

Physique mathématique

QUARTER TURNS AND NEW FACTORIZATIONS OF ROTATIONS

Danail Brezov, Clementina Mladenova*, Ivaïlo Mladenov**

(*Submitted by Academician P. Popivanov on March 26, 2013*)

Abstract

Here we consider new decompositions of the special orthogonal transformations in $\mathbb{R}^3$ into products of two rotations, one of them has a fixed scalar parameter, and the other – a fixed axis. The obtained analytic solutions constitute an alternative parametrization of the group $\mathsf{SO}(3)$ with charts in $\mathbb{S}^2 \times \mathbb{S}^1$. As it should be expected, from topological point of view, this map has singularities – the number of images varies between zero, one, two and infinitely many. The corresponding formulae become particularly simple in the cases involving quarter turns and half turns, although in the latter additional geometric criteria appear. Transferring the same construction to the universal cover $\mathsf{SU}(2) \cong \mathbb{S}^3$ via quaternion parametrization eliminates the problem with infinite scalar parameters. The so obtained map can also be seen as a realization of the Hopf fibration $\mathbb{S}^1 \to \mathbb{S}^3 \to \mathbb{S}^2$.

Key words: decomposition, half turn, quarter turn, quaternion, vector-parameter

2010 Mathematics Subject Classification: 17B81, 22E70, 81R05

1. Introduction. It is well known (Euler's theorem) that all three dimensional rotations have an invariant axis which is unique except in the trivial case (identity transformation) and can be specified by a unit vector $\hat{\mathbf{n}}$. In order to describe the rotation itself, we need an extra parameter, usually chosen to be the angular variable φ. Then we may construct the matrix of the transformation using the famous Rodrigues' formula

$$\mathcal{R}(\hat{\mathbf{n}}, \varphi) = \cos\varphi\, \mathcal{I} + (1 - \cos\varphi)\, \hat{\mathbf{n}} \otimes \hat{\mathbf{n}}^t + \sin\varphi\, \hat{\mathbf{n}}^{\times}, \tag{1}$$

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Equilibrium Configurations of Lipid Bilayer Membranes and Carbon Nanostructures*

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Abstract *The present article concerns the continuum modelling of the mechanical behaviour and equilibrium shapes of two types of nano-scale objects: fluid lipid bilayer membranes and carbon nanostructures. A unified continuum model is used to handle four different case studies. Two of them consist in representing in analytic form cylindrical and axisymmetric equilibrium configurations of single-wall carbon nanotubes and fluid lipid bilayer membranes subjected to uniform hydrostatic pressure. The third one is concerned with determination of possible shapes of junctions between a single-wall carbon nanotube and a flat graphene sheet or another single-wall carbon nanotube. The last one deals with the mechanical behaviour of closed fluid lipid bilayer membranes (vesicles) adhering onto a flat homogeneous rigid substrate subjected to micro-injection and uniform hydrostatic pressure.*

PACS numbers: 61.46.Fg, 81.07.De, 87.16.ad, 87.16.dm, 87.16.D-

Key words: fluid membranes, graphene, carbon nanotubes, carbon nanostructures, junctions, variational statement, Euler–Lagrange equations, natural boundary conditions, cell injection, adhesion, equilibrium shapes

1 Introduction

This article is concerned with the mechanical behaviour and shape analysis of two types of objects of quite different physical and chemical nature — fluid membranes (FM's) and carbon nanostructures (CNS').

Here, by a fluid membrane we mean a membrane formed in aqueous solution by a bilayer of lipid molecules, which are in a fluid state, i.e., the molecules can move freely within the monolayer they belong to. The structure of the bilayer is such that the hydrophobic tails of the lipid molecules situated in different monolayers face one another to form a semi-permeable core, while their hydrophilic heads face the aqueous solutions on either side of the membrane. It is well-known that the lipid bilayer is the main structural component of all biological membranes, the closed lipid bilayer membranes (vesicles) being thought of as the simplest model systems for studying basic physical properties of the more complex biological cells.

By a carbon nanostructure we mean any stable configuration of the curved (bended and/or stretched) graphene such as: carbon nano-tubes (CNT's), nano-horns, nanotori, fullerenes, wormholes, schwartzites and so on. Some of these structures (especially CNT's) are utilized as basic ingredients of nano-structured materials such as nanotube-based nano-composites or functionalized CNT membranes used in water desalination, for instance. Others are basic building blocks of nano-electromechanical systems (NEMS), nano-sensors and other nano-devices.

The underlying idea behind the present contribution is to combine the study of the mechanical behaviour of FM's and CNS' on the bases of a unified continuum mechanics model. In this way, we hope to achieve a significant transfer of knowledge between FM and CNS sciences and thus to accelerate the development of both fields.

The idea for such a unification emerges in a natural way when one compares the known configurations of FM's and CNS' and realizes that their shapes are similar. This similarity is not accidental. It is intimately connected with the following observations:

(a) Regardless of the particular chemical or physical structure, the geometry of both the foregoing types of objects is essentially two-dimensional and therefore it can be described in terms of the differential geometry of surfaces;

(b) Both types of the considered objects exhibit elastic behaviour within a large scale, their elastic properties being characterized (in the simplest models) by a few parameters, and hence one has a good reason to believe that

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VECTOR DECOMPOSITION OF HALF TURNS*

Danail S. Brezov, Clementina D. Mladenova, Ivaïlo M. Mladenov

This study concerns the special case of the symmetric three-dimensional orthogonal matrices and here we suggest a direct method for a representation of any such matrix as a consecutive composition of rotations about three almost arbitrarily chosen axes. Using a suitable representation the task is reduced to solving of a system of quadratic equations for the scalar parameters of the rotations in the decomposition. Then a comparison of the matrix entries selects the actual solutions and dismisses the fake ones. The algorithm is explained in detail and illustrated at the end of the paper via an example.

1. Introduction. The necessity of decomposing the complex rotational motions into two or three successive rotations is dictated by the practical needs of industry and engineering sciences. It is worth to mention that the factorizations of orthogonal matrices play an important role in modern navigation and control of aircrafts, submarines, and communication satellites [8], crystallography and diffractometry [2], or digital image processing [6] and optics [10]. In any of these areas it is necessary to perform several successive displacements in order to obtain the desired setting. Besides, one must construct a framework which is logically and physically free of some constraints because everything happens in the ordinary Euclidean space. It is exactly this fact which explains why the group of the orthogonal matrices in $\mathbb{R}^3$ is somewhat special not only because its various physical applications, but also because for $n = 3$ we have an isomorphism between one-forms and two-forms given by the *Hodge* duality $* : \Lambda^k(\mathbb{R}^n) \to \Lambda^{n-k}(\mathbb{R}^n)$. In this particular case it can be interpreted as a one-to-one correspondence between the vectors and the skew-symmetric matrices in $\mathbb{R}^3$. Since the latter constitute the Lie algebra $\mathfrak{so}(3)$, this allows the standard representation of the rotation generators to be given by the exceptional formula $* : \mathbf{c} \to \mathcal{A}(\mathbf{c}) = \mathbf{c}^{\times}$, or in components $(\mathbf{c}^{\times})_{ij} = \varepsilon_{ilj} c_l$, where ε_{ilj} are the entries of the *Levi-Civita* symbol and the summation is performed over repeated indices following *Einstein* convention. In this way any skew-symmetrical matrix $\mathcal{A}$ can be easily written in the form

$$\mathcal{A}(\mathbf{c}) = \begin{pmatrix} 0 & -c_3 & c_2 \\ c_3 & 0 & -c_1 \\ -c_2 & c_1 & 0 \end{pmatrix}.$$

*2000 Mathematics Subject Classification: 22E15, 22E70, 81R05.
Key words: Euler angles, vector parametrization, factorization.

Some New Results on Three-Dimensional Rotations and Pseudo-Rotations

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Abstract. We use a vector parameter technique to obtain the generalized Euler decompositions with respect to arbitrarily chosen axes for the three-dimensional special orthogonal group $\mathsf{SO}(3)$ and the three-dimensional *Lorentz* group $\mathsf{SO}(2,1)$. Our approach, based on projecting a quaternion (respectively split quaternion) from the corresponding spin cover, has proven quite effective in various problems of geometry and physics [1, 2, 3]. In particular, we obtain explicit (generally double-valued) expressions for the three parameters in the decomposition and discuss separately the degenerate and divergent solutions, as well as decompositions with respect to two axes. There are some straightforward applications of this method in special relativity and quantum mechanics which are discussed elsewhere (see [4]).

Keywords: Quaternions, split quaternions, vector-parameters, Euler decomposition, Rodrigues' formula, hyperbolic geometry
PACS: 02.20.Qs, 02.20.Tw, 06.30.Bp, 06.30.Gv, 45.20.dc, 45.40.Bb

THE EUCLIDEAN CASE

We start with the standard representation of the rotations in the Euclidean space $\mathbb{R}^3$ via the unit quaternions [5, 6, 7]. The special orthogonal group is parameterized by the so-called *vector parameter* (also known as *Rodrigues'* or *Gibbs'* vector (see [1, 8, 9]). The latter appears to be a particularly convenient tool, especially in the context of the generalized Euler decomposition considered here. With its help we obtain quite simple explicit expressions for the scalar (angular) parameters, appropriate for both analytic and numerical use (cf. [10, 11, 12] for comparison).

Quaternions and Vector Parameters

Representing $\mathsf{SU}(2) \cong \mathbb{S}^3$ with unit quaternions leads to a convenient construction for the rotation group via the two-fold projection $\mathsf{SU}(2) \to \mathsf{SO}(3)$. More precisely, we may choose a basis in $\mathfrak{su}(2)$

$$\mathbf{i} = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}, \qquad \mathbf{j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \mathbf{k} = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \tag{1}$$

and introduce unit quaternions as

$$\zeta = \zeta_0 + \zeta_1 \mathbf{i} + \zeta_2 \mathbf{j} + \zeta_3 \mathbf{k}, \qquad |\zeta|^2 = 1,$$

with norm given by

$$|\,\zeta\,|^2 = \frac{1}{2}\mathrm{trace}(\zeta\bar{\zeta}) = \det(\zeta) = \sum_{\mu=0}^{3} \zeta_\mu^2,$$

where $\bar{\zeta} = \zeta_0 - \zeta_1 \mathbf{i} - \zeta_2 \mathbf{j} - \zeta_3 \mathbf{k}$ stands for the *conjugate* quaternion. Note that $|\zeta|^2 = 1$ also implies $\bar{\zeta} = \zeta^{-1}$. Next, we associate with each vector $\mathbf{x} \in \mathbb{R}^3$ a skew-Hermitian matrix

$$\mathbf{x} \to \mathbf{X} = x_1 \mathbf{i} + x_2 \mathbf{j} + x_3 \mathbf{k}, \qquad (\mathbf{x}, \mathbf{x}) = \det \mathbf{X} = x_1^2 + x_2^2 + x_3^2,$$

where x_i are the Cartesian coordinates of the vector $\mathbf{x}$ in the default basis.

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Traveling Wave Solutions of the One-Dimensional Boussinesq Paradigm Equation

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Abstract. The one-dimensional quasi-stationary flow of inviscid liquid in a shallow layer with free surface is described by the so-called Boussinesq Paradigm Equation (BPE). Slightly generalized this equation appears also in the theory of longitudinal vibrations of rods and in the continuum limit for lattices. It is well known that the one-dimensional (1-D) BPE admits a one-parameter family of traveling wave solutions expressed in an analytic form through the "sech" function. In the present contribution, new analytic solutions to the 1-D BPE representing traveling waves are obtained. These solutions are expressed through Weierstrass and Jacobi elliptic functions, which in some cases reduce to elementary functions.

Keywords: Boussinesq Paradigm Equation, traveling wave solutions, Weierstrass and Jacobi elliptic functions
PACS: 02.30.Jr, 02.30.Hq, 02.30.Ik

INTRODUCTION

About 140 years ago Boussinesq [1] studied the fluid flow in the so-called "shallow water" approximation. Considering the shape of the fluid surface, he introduced the fundamental idea that this shape is due to the balance between nonlinearity and dispersion and derived an equation for the case of weakly nonlinear long waves, which can be written in the form

$$u_{tt} = \Delta\left(u - \alpha u^2 + \beta_1 u_{tt} - \beta_2 \Delta u\right) \tag{1}$$

where the function $u(t,x,y)$ describes the shape of the fluid surface, t is the time, x and y are the spatial coordinates

$$u_{tt} = \frac{\partial^2 u}{\partial t^2}, \qquad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

α, β_1 and β_2 are real constants – the amplitude parameter and dispersion coefficients, respectively, which are expressed as

$$\alpha = \frac{3\beta}{2}, \qquad \beta_1 = \frac{\beta}{2}, \qquad \beta_2 = \frac{\beta}{6} \tag{2}$$

in terms of the so-called dispersion parameter β. Nowadays, following Christov [2], Eq. (1) is called Boussinesq Paradigm Equation (BPE).

In one spatial dimension x, Eq. (1), slightly generalized by introducing a new real constant γ, takes the form

$$\frac{\partial^2}{\partial x^2}\left(\gamma u - \alpha u^2 + \beta_1 \frac{\partial^2 u}{\partial t^2} - \beta_2 \frac{\partial^2 u}{\partial x^2}\right) - \frac{\partial^2 u}{\partial t^2} = 0 \tag{3}$$

and is sometimes called the double dispersive equation (DDE), cf. [3]. For $\gamma = 1$, Eq. (3) coincides with the one-dimensional (1-D) BPE.

Let us remark that the double dispersive equation governs the longitudinal strain waves in a free lateral surface rod (see [3, 4] and the references therein). In this case, $u(t,x,y)$ is interpreted as a strain function. Actually, the first attempts to highlight the existence of solitary waves in solid mechanics problems can be traced to the second half of the 20th century. Hutchinson *et al.* [5] considered propagation of longitudinal waves along a straight cylindrical elastic

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VECTOR PARAMETERS IN CLASSICAL HYPERBOLIC GEOMETRY

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Presented by Ivaïlo M. Mladenov

Abstract. Here we use an extension of *Rodrigues' vector parameter* construction for pseudo-rotations in order to obtain explicit formulae for the generalized *Euler* decomposition with arbitrary axes for the structure groups in the classical models of hyperbolic geometry. Although the construction is projected from the universal cover $\mathsf{SU}(1,1) \simeq \mathsf{SL}(2,\mathbb{R})$, most attention is paid to the $2+1$ *Minkowski* space model, following the close analogy with the *Euclidean* case, and various decompositions of the restricted *Lorentz* group $\mathsf{SO}^+(2,1)$ are investigated in detail. At the end we propose some possible applications in *special relativity* and *scattering theory*.

Contents

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Geometry, Integrability and Quantization

CYLINDRICAL FLUID MEMBRANES AND THE EVOLUTIONS OF PLANAR CURVES

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Abstract. An interesting relation between the mKdV equation and the cylindrical equilibrium shapes of fluid membranes is observed. In our setup mKdV arises from the study of the evolution of planar curves in the normal direction.

1. Introduction

This paper unifies and extends the results of two articles, and shows a relation between two problems that appears unrelated.

The first problem comes from the study of equilibrium shapes of fluid membranes. One starts with a functional proposed by Helfrich (see [2], [8]) and studies the corresponding Euler-Lagrange equation. The equilibrium shapes are given as the extrema of the functional

$$= \frac{k_c}{2}\int_S (2H + \quad)^2 \mathrm{d}A + k_G \int_S K \mathrm{d}A + \lambda \int_S \mathrm{d}A + p \int \mathrm{d}V. \qquad (1)$$

Notice that is closely related to the Willmore energy functional. The Euler-Lagrange equation associated with is as follows

$$2k_c \quad {}_S H + k_c(2H + \quad)(2H^2 \qquad H \quad 2K) \quad 2\lambda H + p = \quad . \qquad (2)$$

Here H and K are the mean and Gauss curvatures respectively, k_c and k_G - bending and Gaussian rigidity constants of the membrane, is the **spontaneous curvature** constant, p and λ - the Lagrange multipliers corresponding to the fixed volume and total membrane area and ${}_S$ is the surface Laplacian on the interface of the membrane. The nature of this equation is complex as it is a fourth-order PDE. If

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New Perspective on the Gimbal Lock Problem

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Abstract. We exploit a new technique for obtaining the generalized *Euler* decomposition of three-dimensional rotations [1], based on the vector parameter (also known as *Rodrigues*' or *Gibbs*'s vector) construction in the context of the gimbal lock problem, well known in the applications. From topological point of view the latter may be thought of as a singularity of the parametrization $\pi : \mathbb{RP}^2 \to \mathbb{T}^3$ resulting in the loss of a degree of freedom that is believed to cause troubles in the engineering applications [2], so it is generally considered a problem and takes some effort to avoid. This article, however, focuses on its potential benefits and comments are made on how one can take advantage of them.

Keywords: quaternions, vector parameters, *Euler* decomposition, *Rodrigues*' formula, gimbal lock
PACS: 02.20.Qs, 02.20.Tw, 06.30.Bp, 06.30.Gv, 45.20.dc, 45.40.Bb

THE VECTOR PARAMETER CONSTRUCTION

Representing $\mathsf{SU}(2) \cong \mathbb{S}^3$ with unit quaternions is a standard procedure (see [3]) that leads to the vector parametrization for the rotation group via projection $\mathsf{SU}(2) \to \mathsf{SO}(3)$. More precisely, choosing a basis $\{\mathbf{e}^k\}$ in $\mathfrak{su}(2)$, we introduce the set of unit quaternions[1]

$$\mathsf{SU}(2): \quad \{\zeta = \zeta_0 + \zeta_k \mathbf{e}^k, \quad |\zeta|^2 = 1\}, \qquad \mathbf{e}^i\mathbf{e}^j = -\delta_{ij}\mathbf{e}^0 + \varepsilon_{ijk}\mathbf{e}^k \tag{1}$$

where $\zeta_k \in \mathbb{R}$, $\mathbf{e}^0$ is the identity, δ_{ij} and ε_{ijk} - the symbols of *Kronecker* and *Levi-Civita* respectively, and the norm is given by $|\,\zeta\,|^2 = \zeta_0^2 + \zeta_1^2 + \zeta_2^2 + \zeta_3^2$. Likewise, we associate vectors $\mathbf{x} \in \mathbb{R}^3$ with skew-*Hermitian* matrices $\mathbf{X} = x_k \mathbf{e}^k \in \mathfrak{su}(2)$, or purely imaginary quaternions[2] and let $\mathsf{SU}(2)$ act in its *Lie* algebra via $\mathrm{Ad}_\zeta : \mathbf{X} \to \zeta\,\mathbf{X}\,\zeta^{-1}$, $|\zeta|^2 = 1 \Rightarrow \zeta^{-1} = \bar{\zeta} = (\zeta_o, -\boldsymbol{\zeta})$, which can be viewed as a norm-preserving automorphism of $\mathbb{R}^3$ and the orthogonal matrix, transforming the *Cartesian* coordinates of $\mathbf{x}$ is retrieved as

$$\mathscr{R}(\zeta) = (\zeta_0^2 - \boldsymbol{\zeta}^2)\mathscr{I} + 2\boldsymbol{\zeta}\otimes\boldsymbol{\zeta}^t + 2\zeta_0\,\boldsymbol{\zeta}^\times \tag{2}$$

where $\mathscr{I}$ stands for the identity operator and $\boldsymbol{\zeta}\otimes\boldsymbol{\zeta}^t$ is the tensor (dyadic) product in $\mathbb{R}^3$.

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[1] summation over repeated upper and lower indices is always assumed in the text
[2] we refer to $\boldsymbol{\zeta} \in \mathbb{R}^3$ as the *imaginary*, or *vector* part of $\zeta = (\zeta_0, \boldsymbol{\zeta})$ and ζ_0 - as its *real* or *scalar* part

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Unduloid-like Equilibrium Shapes of Carbon Nanotubes Subjected to Hydrostatic Pressure

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Abstract. The aim of this work is to obtain numerically unduloid-like equilibrium shapes of carbon nanotubes subjected to external pressure.

Mathematics Subject Classification (2010). Primary 82D80; Secondary 74G15, 74G65.

Keywords. Carbon nanotubes, equilibrium shapes, unduloid-like shapes.

1. Introduction

Carbon nanotubes are carbon molecules in the shape of hollow cylindrical fibers of nanometer-size diameter and length-to-diameter ratio of up to 107 : 1. Carbon nanotubes exhibit extraordinary strength, unique electrical properties, and are efficient conductors of heat. For this reason, carbon nanotubes have many practical applications in electronics, optics and other fields of material science. If the tube wall is composed by one layer of carbon atoms, then the tube is referred to as a single-walled one (SWNT). Otherwise, the tube is called multi-walled (MWNT).

The predominating opinion among the scientists working in this field is that they are discovered in 1991 by Sumio Iijima [1]. However, carbon nanotubes have been produced and observed prior to 1991. In 1952 appeared a paper in the Soviet Journal of Physical Chemistry (in Russian) by Radushkevich and Lukyanovich [2] where images of 50 nanometer diameter tubes made of carbon are presented. Oberlin, Endo and Koyama [3] reported observations of hollow carbon fibers (SWNT) with nanometer-scale diameters in 1976. In 1987, Howard G. Tennent of Hyperion Catalysis was issued a US patent for the production of "*... cylindrical discrete carbon fibrils with a constant diameter between about* 3.5 *and about* 70 *nanometers ..., length* 102 *times the diameter ...*"

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CYLINDRICAL HELFRICH SURFACES*

Vladimir Pulov, Mariana Hadzhilazova, Ivaïlo M. Mladenov

The equilibrium shapes of fluid membranes in the spontaneous curvature model (Helfrich's model) are described by the so called Helfrich equation. Surfaces obtained as solutions of the governing equation are called Helfrich surfaces, respectively. By making use of the conformal coordinates and applying Lie group reduction method we construct group-invariant solutions, expressed in terms of the Weierstrass elliptic $\wp$-function. The explicit analytic formulas for the position vector allow to find out the closed directrices generating the Helfrich cylindrical surfaces and to display some of their graphs.

1. Helfrich equation. Membranous biophysical systems, formed in aqueous solution, such as membranes of cells and lipid configurations in living organisms (e.g. lipid vesicles), are generally called *fluid membranes*. Any membrane itself is regarded as a smooth surface $\mathcal{S}$ in the Euclidean space $\mathbb{R}^3$ with the mean and the Gaussian curvatures H and K, respectively. In the spontaneous curvature model (Helfrich's model), widely accepted now, the equilibrium shapes of fluid membranes are determined by solving the *Helfrich equation* [1]

$$\Delta_{\mathcal{S}} H + 2(H^2 + \mathbb{h}H - K)(H - \mathbb{h}) - \frac{2\lambda H}{k} + \frac{p}{k} = 0 \tag{1}$$

which is the Euler-Lagrange equation, obtained in a variational approach by minimizing the free elastic energy functional

$$\mathcal{F} = \frac{k}{2}\int_{\mathcal{S}} (2H + \mathbb{h})^2 \mathrm{d}S + \bar{k}\int_{\mathcal{S}} K \mathrm{d}S$$

under the constraints of fixed enclosed volume and surface area of the membrane.

In the aforementioned formulas the bending rigidity k, the Gaussian rigidity $\bar{k}$, the tensile stress λ, the pressure difference p between the outer and the inner media of the surface (osmotic pressure) and the spontaneous mean curvature $\mathbb{h}$ are the physical characteristics of the membrane, and $\Delta_{\mathcal{S}}$ denotes the Laplace-Beltrami operator on $\mathcal{S}$.

We will call the surfaces obtained as solutions of the Helfrich equation *Helfrich surfaces*.

*2010 Mathematics Subject Classification: 53A05, 74A50, 74K15.
Key words: symmetries, membranes, variational problems, surfaces.

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COVARIANT VECTOR DECOMPOSITION OF THREE-DIMENSIONAL ROTATIONS*

Danail S. Brezov, Clementina D. Mladenova, Ivaïlo M. Mladenov

The main purpose of this paper is to provide an alternative representation for the generalized *Euler* decomposition (with respect to arbitrary axes) obtained in [1, 2] by means of vector parametrization of the *Lie* group $\mathsf{SO}(3)$. The scalar (angular) parameters of the decomposition are explicitly written here as functions depending only on the contravariant components of the compound vector-parameter in the basis, determined by the three axes. We also consider the case of coplanar axes, in which the basis needs to be completed by a third vector and in particular, two-axes decompositions.

1. Vector-parameters in the *Euler* decomposition. Vector-parameters, also known as *Rodrigues'* or *Gibbs'* vectors, are naturally introduced via stereographic projection. For the rotation group in $\mathbb{R}^3$ we consider the spin cover $\mathsf{SU}(2) \cong \mathbb{S}^3 \longrightarrow \mathsf{SO}(3) \cong \mathbb{RP}^3$ and identify $\mathbb{S}^3$ with the set of the unit quaternions (cf. [4])

$$\zeta = (\zeta_0, \boldsymbol{\zeta}) = \zeta_0 + \zeta_1\mathbf{i} + \zeta_2\mathbf{j} + \zeta_3\mathbf{k}, \qquad |\zeta|^2 = \zeta\bar{\zeta} = 1, \qquad \bar{\zeta} = (\zeta_0, -\boldsymbol{\zeta}), \qquad \zeta_\alpha \in \mathbb{R}.$$

The corresponding group morphism is given by the adjoint action of $\mathbb{S}^3$ in its *Lie* algebra of skew-*Hermitian* matrices, in which we expand vectors $\mathbf{x} \in \mathbb{R}^3 \to x_1\mathbf{i}+x_2\mathbf{j}+x_3\mathbf{k} \in \mathfrak{su}(2)$. The resulting $\mathsf{SO}(3)$ matrix transforming the *Cartesian* coordinates of $\mathbf{x}$ has the form

$$\mathcal{R}(\zeta) = (\zeta_0^2 - \boldsymbol{\zeta}^2)\mathcal{I} + 2\,\boldsymbol{\zeta}\otimes\boldsymbol{\zeta}^t + 2\,\zeta_0\boldsymbol{\zeta}^\times, \tag{1}$$

where $\mathcal{I}$ and $\boldsymbol{\zeta}\otimes\boldsymbol{\zeta}^t$ denote the identity and the tensor (dyadic) product in $\mathbb{R}^3$ respectively, whereas $\boldsymbol{\zeta}^\times$ is the skew-symmetric matrix, associated with the vector $\boldsymbol{\zeta}$ via *Hodge* duality. The famous *Rodrigues'* rotation formula then follows directly with the substitution

$$\zeta_0 = \cos\frac{\varphi}{2}, \qquad \boldsymbol{\zeta} = \sin\frac{\varphi}{2}\,\mathbf{n}, \qquad (\mathbf{n},\mathbf{n}) = 1.$$

On the other hand, we may choose to get rid of the unnecessary fourth coordinate by projecting $\zeta \to \mathbf{c} = \dfrac{\boldsymbol{\zeta}}{\zeta_0} = \tan\left(\dfrac{\varphi}{2}\right)\mathbf{n}$ and thus obtain the entries of the rotation matrix (1) expressed as rational functions of the *vector-parameter* $\mathbf{c}$ in the form

$$\mathcal{R}(\mathbf{c}) = \frac{(1-\mathbf{c}^2)\,\mathcal{I} + 2\,\mathbf{c}\otimes\mathbf{c}^t + 2\,\mathbf{c}^\times}{1+\mathbf{c}^2}. \tag{2}$$

*2010 Mathematics Subject Classification: 20C35, 22E70, 81R05.
Key words: quaternions, rotations, Lie group representations.

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ELASTIC STURMIAN SPIRALS

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(*Submitted by Academician P. Popivanov on November 27, 2013*)

Abstract

Curves which belong to the class of the generalized Sturmian spirals and obey to the Elastica equation are studied. Analytical formulae for their parametrizations and a few illustrative plots are presented.

Key words: plane curves, spirals, Euler elastica
2010 Mathematics Subject Classification: 14H50, 53A04, 74K10

1. Introduction. In this paper we study planar curves that represent simultaneously solutions of the Euler elastica problem and generalized Sturmian spirals. Let us start with a brief description of these physical concepts.

The elastic curve minimizes the integral of the curvature squared subject to fixed length and first order initial conditions. Let $\kappa(s)$ be the curvature of a curve parametrized by arc length. We want to minimize the bending energy

$$\int_\gamma \kappa^2(s)\mathrm{d}s \tag{1}$$

among all curves γ with the constraints mentioned above. After solving the variational problem one gets that for planar curves the curvature satisfies the following Euler-Lagrange equation

$$\ddot{\kappa} = -\frac{1}{2}\kappa^3 + \frac{\lambda}{2}\kappa, \tag{2}$$

where λ is a tension constant and the dot(s) denote the derivative(s) with respect to the natural parameter s. An immediate integration yields

$$\dot{\kappa}^2 = -\frac{1}{4}\kappa^4 + \frac{\lambda}{2}\kappa^2 + 2E, \tag{3}$$

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DELAUNAY SURFACES IN TERMS OF WEIERSTRASSIAN FUNCTIONS

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Abstract. Strangely enough (in view of the long time since their original discovery) the description of the Delaunay surfaces via the Weierstrassian functions is absent in the literature. Here we have filled this gap by providing this missing explicit parameterization along with some comments about the alternative parameterization in terms of elliptic integrals.

1. Delaunay Surfaces

Almost two centuries ago the French mathematician Delaunay [3] has classified all surfaces of revolution in $\mathbb{R}^3$ with a constant mean curvature. The respective (and exhaustive) list includes planes, cylinders, spheres, catenoids, unduloids and nodoids. In an Appendix to that paper Sturm characterized these surfaces variation-

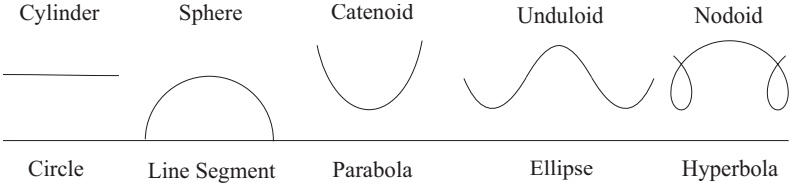

Figure 1: The profile curves of the Delaunay's surfaces obtained by rolling the conics listed below them.