# Review and New Results on Intuitionistic Fuzzy Sets

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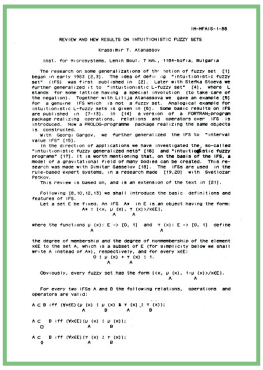
**How to cite:**

Atanassov K. T. Review and New Results on Intuitionistic Fuzzy Sets, Mathematical Foundations of Artificial Intelligence Seminar, Sofia, 1988, Preprint IM-MFAIS-1-88. Reprinted: Int. J. Bioautomation, 2016, 20(S1), S7-S16.



The research on some generalizations of the notion of fuzzy set [1] began in early 1963 [2, 3]. The idea of defining "intuitionistic fuzzy set" (IFS) was first published in [2]. Later with Stefka Stoeva we further generalized it to "intuitionistic $L$-fuzzy set" [4], where $L$ stands for some lattice having a special involution (to take care of the negation). Together with Lilija Atanassova we gave an example [5] for a genuine IFS, which is not a fuzzy set. Analogical example for intuitionistic L-fuzzy sets is given in [6]. Some basic results on IFS are published in [7-13]. In [14] a version of a FORTRAN-program раскаge realizing operations, relations and operators over IFS is introduced. Now a PROLOG-programme package realizing the same objects is constructed.

With Georgi Gargov, we further generalized the IFS to "interval value IFS" [15].

In the direction of applications we have investigated the so-called "intuitionistic fuzzy generalized nets" [16] and "intuitionistic fuzzy programs" [17]. It is worth mentioning that, on the basis of the IFS, a model of a gravitational field of many bodies can be created. This research was made with Dimitar Sasselov [18]. The IFSs are used in the rule-based expert systems, in [19] and [20] with Svetlozar Petkov.

This review is based on, and is an extension of the text in [21].

Following [8, 10, 12, 13] we shall introduce the basic definitions and features of IFS.

Let a set $E$ be fixed. An IFS $A^*$ in $E$ is an object having the form:

$$A^* = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in E\},$$

where the functions $\mu_A(x)$: $E \to [0, 1]$ and $\nu_A(x)$: $E \to [0, 1]$ define the degree of membership and the degree of non-membership of the element $x \in E$ to the set $A$, which is a subset of $E$ (for simplicity below we shall write $A$ instead of $A^*$), respectively, and for every $x \in E$:

$$0 \le \mu_A(x) + \nu_A(x) \le 1.$$

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Obviously, every fuzzy set has the form $\{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle \mid x \in E\}$.

For every two IFSs $A$ and $B$ the following relations, operations and operators are valid:

$A \subset B$ iff $(\forall x \in E)$ $(\mu_A(x) \le \mu_B(x)$ & $\nu_A(x) \ge \nu_B(x))$;

$A \subset_{\Box} B$ iff $(\forall x \in E)$ $(\mu_A(x) \le \mu_B(x))$;

$A \subset_{\Diamond} B$ iff $(\forall x \in E)$ $(\nu_A(x) \ge \nu_B(x))$;

$A \subset B$ iff $(\forall x \in E)$ $(\pi_A(x) \le \pi_B(x))$,

where $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$;

$A = B$ iff $A \subset B$ & $B \subset A$;

$\overline{A} = \{\langle x, \nu_A(x), \mu_A(x)\rangle \mid x \in E\}$;

$A \cap B = \{\langle x, \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x))\rangle \mid x \in E\}$;

$A \cup B = \{\langle x, \max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x))\rangle \mid x \in E\}$;

$A + B = \{\langle x, \mu_A(x) + \mu_B(x) - \mu_A(x).\mu_B(x), \nu_A(x).\nu_B(x))\rangle \mid x \in E\}$;

$A \,.\, B = \{\langle x, \mu_A(x).\mu_B(x), \nu_A(x) + \nu_B(x) - \nu_A(x).\nu_B(x)\rangle \mid x \in E\}$;

$A - B = A \cap \overline{B}$;

$A \times B = \{\langle\langle x, y\rangle, \mu_A(x).\mu_B(x), \nu_A(x).\nu_B(x)\rangle \mid x \in E_1$ & $y \in E_2\}$,

where $E_1$ and $E_2$ are two universums and $A \subset E_1$, $B \subset E_2$.

$\Box A = \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle \mid x \in E\}$;

$\Diamond A = \{\langle x, 1 - \nu_A(x), \nu_A(x)\rangle \mid x \in E\}$;

$CA = \{\langle x, K, L\rangle \mid x \in E\}$,

where $K = \max_{x \in E} \mu_A(\mathrm{x})$, $L = \min_{x \in E} \nu_A(\mathrm{x})$;

$IA = \{\langle x, k, l\rangle \mid x \in E\}$,

where $k = \min_{x \in E} \mu_A(\mathrm{x})$, $l = \max_{x \in E} \nu_A(\mathrm{x})$.

Finally, the operator $M$: $\{X \mid X \subset E$ & $X = \{x, y_1, ..., y_k\}\} \rightarrow E - \{y_1, ..., y_k\}$ is defined:

1) $(\forall x \in E)\ (M(\{x\}) = x)$,

2) $(\forall x, y \in E)\ ((\langle x, \mu_A(x), \nu_A(x)\rangle \in A\ \&\ \langle y, \mu_A(y), \nu_A(y)\rangle \in A \rightarrow$
$$(\mu_A(M(\{x, y\}) = \min(1, \mu_A(x) + \mu_A(y)))$$
$$\&\ (\nu_A(M(\{x, y\}) = \min(\max(1 - \mu_A(x) - \mu_A(y), \nu_A(x), \nu_A(y))),$$

3) $(\forall x, y_1, ..., y_k \in E)\ M(\{x, y_1, ..., y_k\}) = M(\{M(\{x, y_1, ..., y_{k-1}\}), y_k\})$.

The $M$ operator keeps every first element $x \in E$ unchanged, and identifies the second element of $E$ if it exists, with the first element.

Following the fuzzy set from $\alpha$-level idea, a definition is introduced of a set from $(\alpha, \beta)$-level, generated by the IFS A, where $\alpha, \beta \in [0, 1]$ are fixed numbers for which $\alpha + \beta \leq 1$. Formally this set has the form:

$$N_{\alpha,\beta}(A) = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in E\ \&\ \mu_A(x) \geq \alpha\ \&\ \nu_A(x) \leq \beta\}.$$

In a similar way, the set

$$N_\alpha(A) = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in E\ \&\ \mu_A(x) \geq \alpha\},$$

we shall call a set of a level of membership $\alpha$, generated by $A$, and the set

$$N^\alpha(A) = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in E\ \&\ \nu_A(x) \leq \alpha\},$$

we shall call a set of a level of non-membership $\alpha$, generated by $A$.

Let:

$$E_1 = \{N_{\alpha,\beta}(A) \mid A \subset E\ \&\ \alpha, \beta \in [0, 1]\ \&\ \alpha + \beta \leq 1\},$$

$$E_2 = \{N_\alpha(A) \mid A \subset E\ \&\ \alpha \in [0, 1]\},$$

$$E_3 = \{N^\alpha(A) \mid A \subset E\ \&\ \alpha \in [0, 1]\}.$$

It is proved that the classes $E_1$, $E_2$ and $E_3$ are filters.

Let $\alpha \in [0, 1]$ be a fixed number. For the IFS $A$ we shall define the operator $D_\alpha(A)$ through:

$$D_\alpha(A) = \{\langle x, \mu_A(x) + \alpha\pi_A(x), \nu_A(x) + (1 - \alpha)\pi_A(x)\rangle \mid x \in E\}.$$

From this definition it follows that $D_\alpha(A)$ is a fuzzy set.

Let $\alpha, \beta \in [0, 1]$ and $\alpha + \beta \leq 1$. We shall define the operator $F_{\alpha,\beta}(A)$ for the IFS $A$ as:

$$F_{\alpha,\beta}(A) = \{\langle x, \mu_A(x) + \alpha\pi_A(x), \nu_A(x) + \beta\pi_A(x)\rangle \mid x \in E\}.$$

**Theorem 1**: For every IFS $A$ and for every $\alpha, \beta \in [0, 1]$, such that $\alpha + \beta \leq 1$:

(a) $D_0(A) = \Box A$;

(b) $D_1(A) = \Diamond A$;

(c) $D_\alpha(A) = F_{\alpha,1-\alpha}(A)$;

(d) $\overline{F_{\alpha,\beta}(\overline{A})} = F_{\beta,\alpha}(A)$;

(e) $CF_{\alpha,\beta}(A) \subset F_{\alpha,\beta}CA$;

(f) $IF_{\alpha,\beta}(A) \supset F_{\alpha,\beta}IA$.

**Theorem 2**: For every two IFSs $A$ and $B$ and for every $\alpha, \beta \in [0, 1]$, such that $\alpha + \beta \leq 1$:

(a) $F_{\alpha,\beta}(A \cap B) \subset F_{\alpha,\beta}(A) \cap F_{\alpha,\beta}(B)$;

(b) $F_{\alpha,\beta}(A \cup B) \supset F_{\alpha,\beta}(A) \cup F_{\alpha,\beta}(B)$;

(c) $F_{\alpha,\beta}(A + B) \subset F_{\alpha,\beta}(A) + F_{\alpha,\beta}(B)$;

(d) $F_{\alpha,\beta}(A \,.\, B) \supset F_{\alpha,\beta}(A) \,.\, F_{\alpha,\beta}(B)$.

**Theorem 3**: For every IFS $A$ and for every $\alpha, \beta, \Gamma, \delta \in [0, 1]$:

(a) if $\beta + \Gamma \leq 1$, then: $D_\alpha(F_{\beta,\Gamma}(A)) = D_{\alpha+\beta-\alpha\beta-\alpha\Gamma}(A)$;

(b) if $\alpha + \beta \leq 1$, then: $F_{\alpha,\beta}(D_\Gamma(A)) = D_\Gamma(A)$;

(c) if $\alpha + \beta \leq 1$ and $\Gamma + \delta \leq 1$, then: $F_{\alpha,\beta}(F_{\Gamma,\delta}(A)) = F_{\alpha+\Gamma-\alpha\Gamma-\alpha\delta,\ \beta+\delta-\beta\Gamma-\beta\delta}(A)$.

Let $\alpha, \beta \in [0, 1]$. Here (for the first time) we shall define the new operator

$G_{\alpha,\beta}(A) = \{\langle x, \alpha\mu_A(x), \beta\nu_A(x)\rangle \mid x \in E\}$.

Obviously $G_{1,1}(A) = A$ and $G_{0,0}(A) = 0 \equiv \{\langle x, 0, 0\rangle \mid x \in E\}$.

**Theorem 4**: For every IFS $A$ and for every two real numbers $\alpha, \beta \in [0, 1]$:

(a) $G_{\alpha,\beta}$ is an IFS;

(b) if $\alpha \leq \Gamma$, then $G_{\alpha,\beta}(A) \subset G_{\Gamma,\beta}(A)$;

(c) if $\beta \leq \Gamma$, then $G_{\alpha,\beta}(A) \supset G_{\alpha,\Gamma}(A)$;

(d) if $\Gamma$, $\delta \in [0, 1]$, then $G_{\alpha,\beta}(G_{\Gamma,\delta}(A)) = G_{\alpha.\Gamma,\beta.\delta}(A) = G_{\Gamma,\delta}(G_{\alpha,\beta}(A))$;

(e) $G_{\alpha,\beta}(CA) = CG_{\alpha,\beta}(A)$;

(f) $G_{\alpha,\beta}(IA) = IG_{\alpha,\beta}(A)$;

(g) if $\Gamma$, $\delta \in [0, 1]$ and $\Gamma + \delta \leq 1$, then $G_{\alpha,\beta}(F_{\Gamma,\delta}(A)) \subset_{\Box} F_{\Gamma,\delta}(G_{\alpha,\beta}(A)) \subset_{\Diamond} G_{\alpha,\beta}(F_{\Gamma,\delta}(A))$;

(h) $\overline{G_{\alpha,\beta}(\overline{A})} = G_{\beta,\alpha}(A)$.

**Theorem 5**: For every two IFSs $A$ and $B$ and for every $\alpha$, $\beta \in [0, 1]$:

(a) $G_{\alpha,\beta}(A \cap B) = G_{\alpha,\beta}(A) \cap G_{\alpha,\beta}(B)$;

(b) $G_{\alpha,\beta}(A \cup B) = G_{\alpha,\beta}(A) \cup G_{\alpha,\beta}(B)$;

(c) $G_{\alpha,\beta}(A + B) \subset G_{\alpha,\beta}(A) + G_{\alpha,\beta}(B)$;

(d) $G_{\alpha,\beta}(A \,.\, B) \supset G_{\alpha,\beta}(A) \,.\, G_{\alpha,\beta}(B)$.

Following [15, 22] we shall introduce the basic definitions of interval valued IFS (IVIFS).

In [15] it is shown that IFS and interval valued fuzzy sets [23] are equipollent generalizations of the notions of fuzzy sets. But the definition of IFS allows a further generalization.

Let a set $E$ be fixed. An IVIFS $A$ over $E$ is an object having the form:

$A = \{\langle x, M_A(x), N_A(x)\rangle \mid x \in E\}$,

where $M_A(x) \subset [0, 1]$ and $N_A(x) \subset [0, 1]$ are intervals and for every $x \in E$:

$$\sup M_A(x) + \sup N_A(x) \leq 1.$$

For every two IVIFS $A$ and $B$ the following relations, operations and operators are valid (by analogy with above):

$A \subset_{\Box,\inf} B$ iff $(x \in E)$ $(\inf M_A(x) \leq \inf M_B(x))$;

$A \subset_{\Box,\sup} B$ iff $(x \in E)$ $(\sup M_A(x) \leq \sup M_B(x))$;

$A \subset_{\Diamond,\inf} B$ iff $(x \in E)$ $(\inf N_A(x) \geq \inf N_B(x))$;

$A \subset_{\Diamond,\sup} B$ iff $(x \in E)$ $(\sup N_A(x) \geq \sup N_B(x))$;

$A \subset_{\Box} B$ iff $A \subset_{\Box,\inf} B$ & $A \subset_{\Box,\sup} B$;

$A \subset_{\Diamond} B$ iff $A \subset_{\Diamond,\inf} B$ & $A \subset_{\Diamond,\sup} B$;

$A \subset B$ iff $A \subset_{\Box} B$ & $B \subset_{\Box} A$;

$A = B$ iff $A \subset B$ & $B \subset A$;

$\overline{A} = \{\langle x, N_A(x), M_A(x)\rangle \mid x \in E\}$;

$A \cap B = \{\langle x, [\min(\inf M_A(x), \inf M_B(x)), \min(\sup M_A(x), \sup M_B(x))],$
$[\max(\inf N_A(x), \inf N_B(x)), \max(\sup N_A(x), \sup N_B(x))]\rangle \mid x \in E\}$;

$A \cup B = \{\langle x, [\max(\inf M_A(x), \inf M_B(x)), \max(\sup M_A(x), \sup M_B(x))],$
$[\min(\inf N_A(x), \inf N_B(x)), \min(\sup N_A(x), \sup N_B(x))]\rangle \mid x \in E\}$;

$A + B = \{\langle x, [\inf M_A(x) + \inf M_B(x) - \inf M_A(x) . \inf M_B(x), \sup M_A(x) + \sup M_B(x) -$
$- \sup M_A(x) . \sup M_B(x)], [\inf N_A(x) . \inf N_B(x), \sup N_A(x) . \sup N_B(x)]\rangle \mid x \in E\}$;

$A . B = \{\langle x, [\inf M_A(x) . \inf M_B(x), \sup M_A(x) . \sup M_B(x)], [\inf N_A(x) + \inf N_B(x) -$
$- \inf N_A(x) . \inf N_B(x), \sup N_A(x) + \sup N_B(x) - \sup N_A(x) . \sup N_B(x)]\rangle \mid x \in E\}$;

$\Box A = \{\langle x, M_A(x), [\inf N_A(x), 1 - \sup M_A(x)]\rangle \mid x \in E\}$;

$\Diamond A = \{\langle x, [\inf M_A(x), 1 - \sup N_A(x)], N_A(x)\rangle \mid x \in E\}$.

Modifications of the above theorems are valid for IVIFSs.

An operator which associates every IVIFS with an IFS can be defined. Let $A$ be an IVIFS. Then:

$$*A = \{\langle x, \inf M_A(x), \inf N_A(x)\rangle \mid x \in E\}.$$

On the other hand, let $A$ be an IFS. Then the following operators can be defined:

$\#_1(A) = \{\mathrm{B} \mid \mathrm{B} = \{\langle x, M_B(x), N_B(x)\rangle \mid x \in E\}$ & $(\forall x \in E)$ $(\sup M_B(x) + \sup N_B(x) \le 1)$ &
$(\forall x \in E)$ $(\inf M_B(x) \ge \mu_A(x)$ & $\sup N_B(x) \le \nu_A(x))\}$;

$\#_2(A) = \{\mathrm{B} \mid \mathrm{B} = \{\langle x, M_B(x), N_B(x)\rangle \mid x \in E\}$ & $(\forall x \in E)$ $(\sup M_B(x) + \sup N_B(x) \le 1)$ &
$(\forall x \in E)$ $(\sup M_B(x) \le \mu_A(x)$ & $\inf N_B(x) \ge \nu_A(x))\}$.

**Theorem 6**: For every IFS $A$:

(a) $\#_1(A) = \{B \mid A \subset *B\}$;

(b) $\#_2(A) = \{B \mid *B \subset A\}$.

Let $\alpha, \beta \in [0, 1]$ and $0 \le \alpha + \beta \le 1$. For the IVIFS $A$ we shall define:

$D_\alpha(A) = \{\langle x, [\inf M_A(x), \sup M_A(x) + \alpha(1 - \sup M_A(x) - \sup N_A(x))],$

$[\inf N_A(x), \sup N_A(x) + (1 - \alpha).(1 - \sup M_A(x) - \sup N_A(x))]\rangle \mid x \in E\}.$

$$F_{\alpha,\beta}(A) = \{\langle x, [\inf M_A(x), \sup M_A(x) + \alpha(1 - \sup M_A(x) - \sup N_A(x))],$$
$$[\inf N_A(x), \sup N_A(x) + \beta.(1 - \sup M_A(x) - \sup N_A(x))]\rangle \mid x \in E\}.$$

Modifications of the respective theorems above are valid for IFIVSs.

Finally we shall note that norms and metrics on IFSs are defined in [24, 25].

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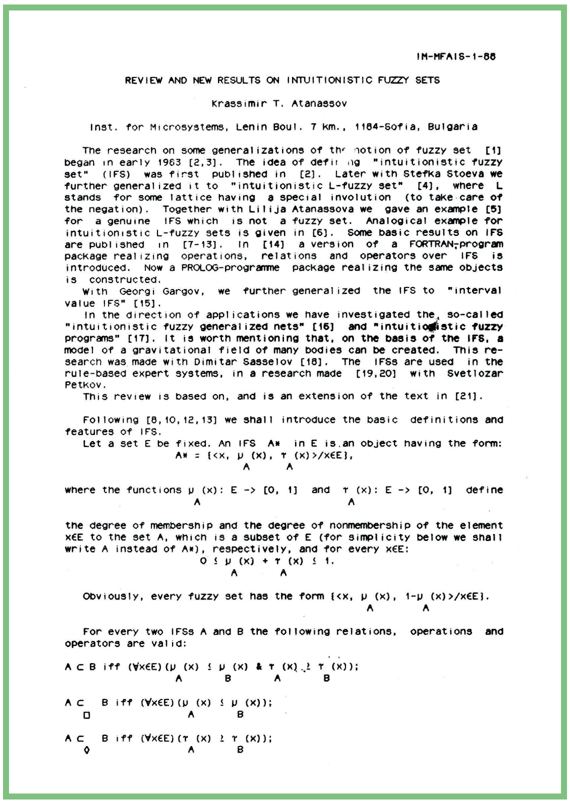

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REVIEW AND NEW RESULTS ON INTUITIONISTIC FUZZY SETS

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The research on some generalizations of the notion of fuzzy set [1] began in early 1983 [2,3]. The idea of defining "intuitionistic fuzzy set" (IFS) was first published in [2]. Later with Stefka Stoeva we further generalized it to "intuitionistic L-fuzzy set" [4], where L stands for some lattice having a special involution (to take care of the negation). Together with Lilija Atanassova we gave an example [5] for a genuine IFS which is not a fuzzy set. Analogical example for intuitionistic L-fuzzy sets is given in [6]. Some basic results on IFS are published in [7-13]. In [14] a version of a FORTRAN-program package realizing operations, relations and operators over IFS is introduced. Now a PROLOG-programme package realizing the same objects is constructed.

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In the direction of applications we have investigated the so-called "intuitionistic fuzzy generalized nets" [16] and "intuitionistic fuzzy programs" [17]. It is worth mentioning that, on the basis of the IFS, a model of a gravitational field of many bodies can be created. This research was made with Dimitar Sasselov [18]. The IFSs are used in the rule-based expert systems, in a research made [19,20] with Svetlozar Petkov.

This review is based on, and is an extension of the text in [21].

Following [8,10,12,13] we shall introduce the basic definitions and features of IFS.

Let a set E be fixed. An IFS A* in E is an object having the form:

$$A^* = \{\langle x, \mu_A(x), \gamma_A(x)\rangle / x \in E\},$$

where the functions $\mu_A(x): E \to [0, 1]$ and $\gamma_A(x): E \to [0, 1]$ define the degree of membership and the degree of nonmembership of the element $x \in E$ to the set A, which is a subset of E (for simplicity below we shall write A instead of A*), respectively, and for every $x \in E$:

$$0 \le \mu_A(x) + \gamma_A(x) \le 1.$$

Obviously, every fuzzy set has the form $\{\langle x, \mu_A(x), 1-\mu_A(x)\rangle / x \in E\}$.

For every two IFSs A and B the following relations, operations and operators are valid:

$A \subset B$ iff $(\forall x \in E)(\mu_A(x) \le \mu_B(x)\ \&\ \gamma_A(x) \ge \gamma_B(x))$;

$A \subset_{\Box} B$ iff $(\forall x \in E)(\mu_A(x) \le \mu_B(x))$;

$A \subset_{\Diamond} B$ iff $(\forall x \in E)(\gamma_A(x) \ge \gamma_B(x))$;

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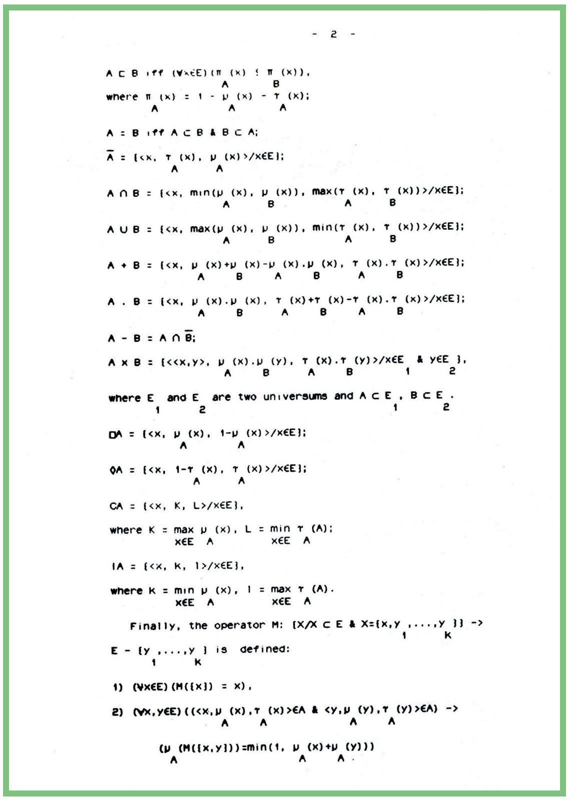

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$A \subset B$ iff $(\forall x \in E)(\pi_A(x) \le \pi_B(x))$,

where $\pi_A(x) = 1 - \mu_A(x) - \gamma_A(x)$;

$A = B$ iff $A \subset B\ \&\ B \subset A$;

$\overline{A} = \{\langle x, \gamma_A(x), \mu_A(x)\rangle / x \in E\}$;

$A \cap B = \{\langle x, \min(\mu_A(x), \mu_B(x)), \max(\gamma_A(x), \gamma_B(x))\rangle / x \in E\}$;

$A \cup B = \{\langle x, \max(\mu_A(x), \mu_B(x)), \min(\gamma_A(x), \gamma_B(x))\rangle / x \in E\}$;

$A + B = \{\langle x, \mu_A(x)+\mu_B(x)-\mu_A(x).\mu_B(x), \gamma_A(x).\gamma_B(x)\rangle / x \in E\}$;

$A\ .\ B = \{\langle x, \mu_A(x).\mu_B(x), \gamma_A(x)+\gamma_B(x)-\gamma_A(x).\gamma_B(x)\rangle / x \in E\}$;

$A - B = A \cap \overline{B}$;

$A \times B = \{\langle\langle x,y\rangle, \mu_A(x).\mu_B(y), \gamma_A(x).\gamma_B(y)\rangle / x \in E_1\ \&\ y \in E_2\}$,

where $E_1$ and $E_2$ are two universums and $A \subset E_1$, $B \subset E_2$.

$\Box A = \{\langle x, \mu_A(x), 1-\mu_A(x)\rangle / x \in E\}$;

$\Diamond A = \{\langle x, 1-\gamma_A(x), \gamma_A(x)\rangle / x \in E\}$;

$CA = \{\langle x, K, L\rangle / x \in E\}$,

where $K = \max_{x \in E} \mu_A(x)$, $L = \min_{x \in E} \gamma_A(A)$;

$IA = \{\langle x, k, l\rangle / x \in E\}$,

where $k = \min_{x \in E} \mu_A(x)$, $l = \max_{x \in E} \gamma_A(A)$.

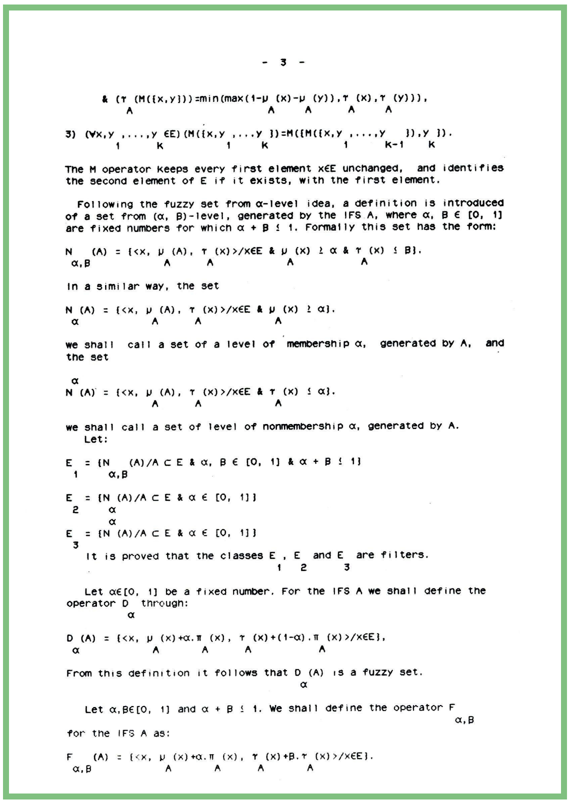
Finally, the operator $M: \{X/X \subset E\ \&\ X=\{x,y_1,\dots,y_k\}\} \to E - \{y_1,\dots,y_k\}$ is defined:

1) $(\forall x \in E)(M(\{x\}) = x)$,

2) $(\forall x,y \in E)((\langle x,\mu_A(x),\gamma_A(x)\rangle \in A\ \&\ \langle y,\mu_A(y),\gamma_A(y)\rangle \in A) \to$

$(\mu_A(M(\{x,y\}))=\min(1, \mu_A(x)+\mu_A(y)))$

Page 2



\- 3 -

$\&\ (\gamma_A(M(\{x,y\}))=\min(\max(1-\mu_A(x)-\mu_A(y)),\gamma_A(x),\gamma_A(y)))$,

3) $(\forall x,y_1,\dots,y_k \in E)(M(\{x,y_1,\dots,y_k\})=M(\{M(\{x,y_1,\dots,y_{k-1}\}),y_k\}))$.

The M operator keeps every first element $x \in E$ unchanged, and identifies the second element of E if it exists, with the first element.

Following the fuzzy set from α-level idea, a definition is introduced of a set from (α, β)-level, generated by the IFS A, where α, β ∈ [0, 1] are fixed numbers for which α + β ≤ 1. Formally this set has the form:

$N_{\alpha,\beta}(A) = \{\langle x, \mu_A(A), \gamma_A(x)\rangle / x \in E\ \&\ \mu_A(x) \ge \alpha\ \&\ \gamma_A(x) \le \beta\}$.

In a similar way, the set

$N_\alpha(A) = \{\langle x, \mu_A(A), \gamma_A(x)\rangle / x \in E\ \&\ \mu_A(x) \ge \alpha\}$.

we shall call a set of a level of membership α, generated by A, and the set

$N^\alpha(A) = \{\langle x, \mu_A(A), \gamma_A(x)\rangle / x \in E\ \&\ \gamma_A(x) \le \alpha\}$.

we shall call a set of level of nonmembership α, generated by A.

Let:

$E_1 = \{N_{\alpha,\beta}(A)/A \subset E\ \&\ \alpha, \beta \in [0, 1]\ \&\ \alpha + \beta \le 1\}$

$E_2 = \{N_\alpha(A)/A \subset E\ \&\ \alpha \in [0, 1]\}$

$E_3 = \{N^\alpha(A)/A \subset E\ \&\ \alpha \in [0, 1]\}$

It is proved that the classes $E_1$, $E_2$ and $E_3$ are filters.

Let α∈[0, 1] be a fixed number. For the IFS A we shall define the operator $D_\alpha$ through:

$D_\alpha(A) = \{\langle x, \mu_A(x)+\alpha.\pi_A(x), \gamma_A(x)+(1-\alpha).\pi_A(x)\rangle / x \in E\}$,

From this definition it follows that $D_\alpha(A)$ is a fuzzy set.

Let α,β∈[0, 1] and α + β ≤ 1. We shall define the operator $F_{\alpha,\beta}$ for the IFS A as:

$F_{\alpha,\beta}(A) = \{\langle x, \mu_A(x)+\alpha.\pi_A(x), \gamma_A(x)+\beta.\gamma_A(x)\rangle / x \in E\}$.

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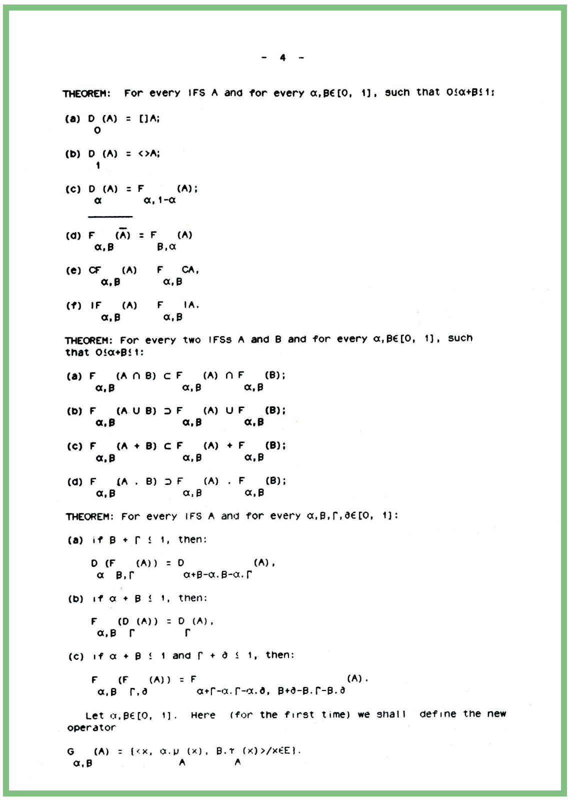

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THEOREM: For every IFS A and for every α,β∈[0, 1], such that 0≤α+β≤1:

(a) $D_0(A) = \Box A$;

(b) $D_1(A) = \Diamond A$;

(c) $D_\alpha(A) = F_{\alpha,1-\alpha}(A)$;

(d) $\overline{F_{\alpha,\beta}(\overline{A})} = F_{\beta,\alpha}(A)$

(e) $CF_{\alpha,\beta}(A)\quad F_{\alpha,\beta}CA$,

(f) $IF_{\alpha,\beta}(A)\quad F_{\alpha,\beta}IA$.

THEOREM: For every two IFSs A and B and for every α,β∈[0, 1], such that 0≤α+β≤1:

(a) $F_{\alpha,\beta}(A \cap B) \subset F_{\alpha,\beta}(A) \cap F_{\alpha,\beta}(B)$;

(b) $F_{\alpha,\beta}(A \cup B) \supset F_{\alpha,\beta}(A) \cup F_{\alpha,\beta}(B)$;

(c) $F_{\alpha,\beta}(A + B) \subset F_{\alpha,\beta}(A) + F_{\alpha,\beta}(B)$;

(d) $F_{\alpha,\beta}(A\ .\ B) \supset F_{\alpha,\beta}(A)\ .\ F_{\alpha,\beta}(B)$;

THEOREM: For every IFS A and for every α,β,γ,δ∈[0, 1]:

(a) if β + γ ≤ 1, then:

$D_\alpha(F_{\beta,\gamma}(A)) = D_{\alpha+\beta-\alpha.\beta-\alpha.\gamma}(A)$,

(b) if α + β ≤ 1, then:

$F_{\alpha,\beta}(D_\gamma(A)) = D_\gamma(A)$,

(c) if α + β ≤ 1 and γ + δ ≤ 1, then:

$F_{\alpha,\beta}(F_{\gamma,\delta}(A)) = F_{\alpha+\gamma-\alpha.\gamma-\alpha.\delta,\ \beta+\delta-\beta.\gamma-\beta.\delta}(A)$.

Let α,β∈[0, 1]. Here (for the first time) we shall define the new operator

$G_{\alpha,\beta}(A) = \{\langle x, \alpha.\mu_A(x), \beta.\gamma_A(x)\rangle / x \in E\}$.

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Obviously $G_{1,1}(A) = A$ and $G_{0,0}(A) = O \equiv \{\langle x,0,0\rangle / x\in E\}$

THEOREM: For every IFS A and for every two real numbers $\alpha,\beta\in[0, 1]$:

(a) $G_{\alpha,\beta}$ is an IFS;

(b) if $\alpha \le \Gamma$, then $G_{\alpha,\beta}(A) \subset G_{\Gamma,\beta}(A)$;

(c) if $\beta \le \Gamma$, then $G_{\alpha,\beta}(A) \supset G_{\alpha,\Gamma}(A)$;

(d) if $\Gamma,\partial\in[0, 1]$, then

$$G_{\alpha,\beta}(G_{\Gamma,\partial}(A)) = G_{\alpha.\Gamma,\beta.\partial}(A) = G_{\Gamma,\partial}(G_{\alpha,\beta}(A));$$

(e) $G_{\alpha,\beta}(CA) = CG_{\alpha,\beta}(A)$;

(f) $G_{\alpha,\beta}(IA) = IG_{\alpha,\beta}(A)$;

(g) if $\Gamma,\partial\in[0, 1]$ and $\Gamma + \partial \le 1$, then

$$G_{\alpha,\beta}(F_{\Gamma,\partial}(A)) \subset_{\Box} F_{\Gamma,\partial}(G_{\alpha,\beta}(A)) \subset_{\Diamond} G_{\alpha,\beta}(F_{\Gamma,\partial}(A));$$

---

(h) $G_{\alpha,\beta}(\overline{A}) = G_{\beta,\alpha}(A)$.

THEOREM: For every two IFSs A and B and for every $\alpha,\beta\in[0, 1]$:

(a) $G_{\alpha,\beta}(A \cap B) = G_{\alpha,\beta}(A) \cap G_{\alpha,\beta}(B)$;

(b) $G_{\alpha,\beta}(A \cup B) = G_{\alpha,\beta}(A) \cup G_{\alpha,\beta}(B)$;

(c) $G_{\alpha,\beta}(A + B) \subset G_{\alpha,\beta}(A) + G_{\alpha,\beta}(B)$;

(d) $G_{\alpha,\beta}(A \,.\, B) \supset G_{\alpha,\beta}(A) \,.\, G_{\alpha,\beta}(B)$;

Following [15,22] we shall introduce the basic definitions of interval valued IFS (IVIFS).

In [15] it is shown that IFS and interval valued fuzzy sets [23] are equipollent generalizations of the notion of fuzzy sets. But the definition of IFS allows a further generalization.

Let a set E be fixed. An IVIFS A over E is an object having the form:

$$A = \{\langle x, M_A(x), N_A(x)\rangle / x\in E\},$$

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where $M_A(x) \subset [0,1]$ and $N_A(x) \subset [0,1]$ are intervals and for every $x\in E$:

$$supM_A(x) + supN_A(x) \le 1.$$

For every two IVIFSs A and B the following relations, operations and operators are valid (by analogy with above):

$A \subset_{\Box,inf} B$ iff $(\ x\in E)(infM_A(x) \le infM_B(x))$;

$A \subset_{\Box,sup} B$ iff $(\ x\in E)(supM_A(x) \le supM_B(x))$;

$A \subset_{\Diamond,inf} B$ iff $(\ x\in E)(infN_A(x) \ge infN_B(x))$;

$A \subset_{\Diamond,sup} B$ iff $(\ x\in E)(supN_A(x) \ge supN_B(x))$;

$A \subset_{\Box} B$ iff $A \subset_{\Box,inf} B$ & $A \subset_{\Box,sup} B$;

$A \subset_{\Diamond} B$ iff $A \subset_{\Diamond,inf} B$ & $A \subset_{\Diamond,sup} B$;

$A \subset B$ iff $A \subset_{\Box} B$ & $B \subset_{\Diamond} A$;

$A = B$ iff $A \subset B$ & $B \subset A$;

$\overline{A} = \{\langle x, N_A(x), M_A(x)\rangle / x\in E\}$;

$$A \cap B = \{\langle x, [min(infM_A(x), infM_B(x)), min(supM_A(x), supM_B(x))],$$
$$[max(infN_A(x), infN_B(x)), max(supN_A(x), supN_B(x))]\rangle / x\in E\};$$

$$A \cup B = \{\langle x, [max(infM_A(x), infM_B(x)), max(supM_A(x), supM_B(x))],$$
$$[min(infN_A(x), infN_B(x)), min(supN_A(x), supN_B(x))]\rangle / x\in E\};$$

$$A + B = \{\langle x, [infM_A(x)+infM_B(x)-infM_A(x).infM_B(x), supM_A(x)+supM_B(x)-$$
$$supM_A(x).supM_B(x)], [infN_A(x).infN_B(x), supN_A(x).supN_B(x)]\rangle / x\in E\};$$

$$A \,.\, B = \{\langle x, [infM_A(x).infM_B(x), supM_A(x).supM_B(x)], [infN_A(x)+infN_B(x)$$
$$-infN_A(x).infN_B(x), supN_A(x)+supN_B(x)-supN_A(x).supN_B(x)]\rangle / x\in E\};$$

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$$\Box A = \{\langle x, M_A(x), [infN_A(x), 1-supM_A(x)]\rangle / x\in E\};$$

$$\Diamond A = \{\langle x, [infM_A(x), 1-supN_A(x)], N_A(x)\rangle / x\in E\}.$$

Modifications of the above theorems are valid for IVIFSs.

An operator which associates every IVIFS with an IFS can be defined. Let A be an IVIFS. Then:

$$*A = \{\langle x, infM_A(x), infN_A(x)\rangle / x\in E\}.$$

On the other hand, let A be an IFS. Then the following operators can be defined:

$$\#_1(A) = \{B / B = \{\langle x, M_B(x), N_B(x)\rangle : x\in E\} \ \& \ (\forall x\in E)(supM_B(x)+$$
$$supN_B(x) \le 1) \ \& \ (\ x\in E)(infM_B(x) \ge \mu_A(x) \ \& \ supN_B(x) \le \tau_A(x))\}$$

$$\#_2(A) = \{B / B = \{\langle x, M_B(x), N_B(x)\rangle : x\in E\} \ \& \ (\forall x\in E)(supM_B(x)+$$
$$supN_B(x) \le 1) \ \& \ (\ x\in E)(supM_B(x) \le \mu_A(x) \ \& \ infN_B(x) \ge \tau_A(x))\}$$

THEOREM: For every IFS A:

(a) $\#_1(A) = \{B / A \subset *B\}$.

(b) $\#_2(A) = \{B / *B \subset A\}$.

Let $\alpha,\beta\in[0, 1]$ and $0 \le \alpha + \beta \le 1$. For the IVIFS A we shall define:

$$D_\alpha(A) = \{\langle x, [infM_A(x), supM_A(x)+\alpha.(1-supM_A(x)-supN_A(x))],$$
$$[infN_A(x), supN_A(x)+(1-\alpha).(1-supM_A(x)-supN_A(x))]\rangle : x \in E\},$$

$$F_{\alpha,\beta}(A) = \{\langle x, [infM_A(x), supM_A(x)+\alpha.(1-supM_A(x)-supN_A(x))$$
$$[infN_A(x), supN_A(x)+\beta.(1-supM_A(x)-supN_A(x))]\rangle : x \in E\}.$$

Modifications of the respective theorems above are valid for IVIFSs.

Finally we shall note that norms and metricses on IFSs are defined in [24,25].

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