Two Variants of Intuitionistic Fuzzy Propositional Calculus

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The definition of fuzzy set (FS) is the basis for defining fuzzy propositional calculus (e.g. see [1]). Here we shall construct two variants of intuitionistic fuzzy propositional calculus (IFPC), basing our construction on the definition of intuitionistic fuzzy sets (IFS) [2] which are an extension of the FS and using the notations from the theory of propositional calculus after [3].

To each proposition (in the classical meaning) one can assign its truth value: truth – denoted by 1, or falsum – 0. In the case of fuzzy logics this truth value is a real number in the interval [0, 1] and can be called "truth degree" of a particular proposition. Here we add one more value – "falsum degree" – which will be in the interval [0, 1] as well. Thus one assigns to the proposition p two real numbers $\mu(p)$ and $\nu(p)$ moreover the constraint is valid:

$$\mu(p) + \nu(p) \leq 1.$$

Let this be done by an evaluation function V defined such that:

$$V(p) = \langle \mu(p), \nu(p) \rangle.$$

Hence the function $V : S \rightarrow [0, 1] \times [0, 1]$ gives the truth and falsum degrees from the class of all propositions.

The evaluation function V can be defined in different ways.

We assume that the evaluation function V is defined so that it assigns to the logical truth T:

$$V(T) = \langle 1, 0 \rangle,$$

or to the logical falsum F:

$$V(F) = \langle 0, 1 \rangle.$$

We shall discuss below the truth and falsum degrees of propositions which result from the application of logical operations (unary and binary) over output propositions which have known values of its evaluation function. The negation $\neg p$ of the proposition p will be defined through

$$V(\neg p) = \langle \nu(p), \mu(p) \rangle.$$

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When

$$V(p) = 1 - \mu(p),$$

i.e.,

$$V(p) = \langle \mu(p), 1 - \mu(p) \rangle,$$

for $\neg p$ we get:

$$V(\neg p) = \langle 1 - \mu(p), \mu(p) \rangle,$$

which coincides with the result from [1].

Depending on the way of definition of the operation "$\supset$", can be obtained different variants of IFPC.

1. sg-variant of IFPC

When the values $V(p)$ and $V(q)$ of the propositions p and q are known, the evaluation function V can be extended by its definition also for operations "&", "$\vee$" and "$\supset$" through:

$$V(p \,\&\, q) = \langle \min(\mu(p), \mu(q)), \max(\nu(p), \nu(q)) \rangle,$$

$$V(p \vee q) = \langle \max\, (\mu(p), \mu(q)), \min(\nu(p), \nu(q)) \rangle,$$

$$V(p \supset q) = \langle 1 - (1 - \mu(q)).\mathrm{sg}(\mu(p) - \mu(q)), \nu(q).\mathrm{sg}(\mu(p) - \mu(q)).\mathrm{sg}(\nu(q) - \nu(p)) \rangle,$$

where

$$\mathrm{sg}(x) = 1, \text{ if } x > 0, \text{ and } \mathrm{sg}(x) = 0, \text{ if } x \leq 0.$$

The first two definitions transferred in the classical and fuzzy cases, coincide entirely with the corresponding definitions there. Although the definition of "$\supset$" is more complex, the same is valid for it too: when $p, q \in \{F, T\}$ the function V has values:

p	$V(p)$	q	$V(q)$	$V(p \supset q)$
F	$\langle 0, 1 \rangle$	F	$\langle 0, 1 \rangle$	$\langle 1, 0 \rangle$
F	$\langle 0, 1 \rangle$	T	$\langle 1, 0 \rangle$	$\langle 1, 0 \rangle$
T	$\langle 1, 0 \rangle$	F	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$
T	$\langle 1, 0 \rangle$	T	$\langle 1, 0 \rangle$	$\langle 1, 0 \rangle$

By analogy with the operations over IFS from [2] it will be convenient to define for the propositions $p, q \in S$:

$$\neg V(p) = V(\neg p),$$

$$V(p) \wedge V(q) = V(p \,\&\, q),$$

$$V(p) \vee V(q) = V(p \vee q),$$

$$V(p) \rightarrow V(q) = V(p \supset q).$$

A given propositional form A (c.f. [3]: *each proposition is a propositional form; if A is a propositional form then $\neg A$ is a pro-positional form; if A and B are propositional forms, then A & B, A $\vee$ B, A $\supset$ B are propositional forms*) will be called a tautology if

$$V(A) = \langle 1, 0 \rangle.$$

Theorem 1: If A and $A \supset B$ are tautologies, then B is also a tautology.
Proof: Once A and $A \supset B$ are tautologies then

$$V(A) = V(A \supset B) = \langle 1, 0 \rangle,$$

i.e.,

$$\mu(A) = 1,$$
$$\nu(A) = 0,$$
$$\mu(A \supset B) = 1 - (1 - \mu(B)).\text{sg}(\mu(A) - \mu(B)) = 1,$$
$$\nu(A \supset B) = \nu(B).\text{sg}(\mu(A) - \mu(B)).\text{sg}(\nu(B) - \nu(A)) = 0.$$

Hence:

$$1 - \mu(B) = 0 \quad \text{or} \quad \text{sg}(1 - \mu(B)) = 0.$$

and in the same time

$$\nu(B) = 0 \quad \text{or} \quad \text{sg}(\mu(A) - \mu(B)) = 0 \quad \text{or} \quad \text{sg}(\nu(B) - \nu(A)) = 0,$$

but

$$1 - \mu(B) = \text{sg}(1 - \mu(B)) = 0$$

exactly then when

$$\mu(A) = 1,$$

from where it follows directly that

$$\nu(B) = 0,$$

i.e., B is a tautology. □

Theorem 2: If A, B and C are tautologies then:

(a) $A \supset A$,

(b) $A \supset (B \supset A)$,

(c) $A \mathbin{\&} B \supset A$,

(d) $A \mathbin{\&} B \supset B$,

(e) $A \supset (A \vee B)$,

(f) $B \supset (A \vee B)$,

(g) $A \supset (B \supset (A \mathbin{\&} B))$,

(h) $(A \supset C) \supset ((B \supset C) \supset ((A \vee B) \supset C))$,

(i) $\neg\neg A \supset A$,

(j) $(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$

are tautologies.

Proof: Let consider everywhere then

$$V(A) = \langle a, b \rangle,$$

$$V(B) = \langle c, d \rangle,$$

$$V(C) = \langle e, f\rangle.$$

From the definitions above we obtain consequently:

(a) $V(A \supset A) = V(A) \to V(A)$

$= \langle 1 - (1 - a).\text{sg}(a - a), b.\text{sg}(a - a).\text{sg}(b - b)\rangle$

$= \langle 1, 0\rangle$.

(b) $V(A \supset (B \supset A)) = V(A) \to (V(B) \to V(A))$

$= \langle a, b\rangle \to \langle 1 - (1 - a).\text{sg}(c - a), b.\text{sg}(c - a).\text{sg}(b - d)\rangle$

$= \langle 1 - (1 - (1 - (1 - a).\text{sg}(c - a))).\text{sg}(a - 1 + (1 - a).\text{sg}(c - a)),$
$b.\text{sg}(c - a).\text{sg}(b - d).\text{sg}(a - 1 + (1 - a).\text{sg}(c - a)).\text{sg}(b.\text{sg}(c - a).\text{sg}(b - d) - b)\rangle$

if $a \geq c$:

$= \langle 1, 0\rangle$;

if $a <$ c:

$= \langle l - (l - a).\text{sg}(a - 1 + (1 - a).1), b.\text{sg}(b - d).\text{sg}(a - 1 + (1 - a).1).\text{sg}(b.\text{sg}(b - d) - b)$

$= \langle 1, 0\rangle$.

(c) $V(A \,\&\, B \supset A) = \langle \min(a, c), \max(b, d)\rangle \to \langle a, b\rangle$

$= \langle 1 - (l - a).\text{sg}(\min(a, c) - a), b.\text{sg}(\min(a, c) - a).\text{sg}(b - \max(b, d))\rangle$

$= \langle 1, 0\rangle$.

(d) is proved analogically.

(e) $V(A \vee B \supset A) = \langle a, b\rangle \to \langle \max(a, c), \min(b, d)\rangle$

$= \langle 1 - (l - \max(a, c)).\text{sg}(a - \max(a, c)), \min(b, d).\text{sg}(a - \max(a, c)).\text{sg}(\min(b, d) - b)\rangle$

$= \langle 1, 0\rangle$.

(f) is proved analogically.

(g) $V(A \supset (B \supset (A \,\&\, B)))$

$= \langle a, b\rangle \to (\langle c, d\rangle \to \langle \min(a, c), \max(b, d)\rangle)$

$= \langle a, b\rangle \to \langle 1 - (1 - \min(a, c)).\text{sg}(c - \min(a, c)),$
$\max(b, d).\text{sg}(c - \min(a, c).\text{sg}(\max(b, d) - d)\rangle$

$= \langle 1 - (1 - \min(a, c)).\text{sg}(c - \min(a, c)).\text{sg}(a - 1 + (l - \min(a, c)).\text{sg}(c + \min(a, c)),$
$\max(b, d).\text{sg}(c - \min(a, c)).\text{sg}(\max(b, d) - d).\text{sg}(a - 1 + \min(a, c)).\text{sg}(c - \min(a, c)))$
$.\text{sg}(\max(b, d).\text{sg}(c - \min(a, c)).\text{sg}(\max(b, d) - d) - b)\rangle$

(from: $a - 1 + (1 - \min(a, c)).\mathrm{sg}(c - \min(a, c)) \le a - 1 + (1 - a).1 = 0$)

$= \langle 1, 0\rangle$.

(h) $V((A \supset C) \supset ((B \supset C) \supset ((A \vee B) \supset C)))$

$= (\langle a, b\rangle \to \langle e, f\rangle) \to ((\langle c, d\rangle \to \langle e, f\rangle) \to (\langle \max(a, c), \min(b, d)\rangle \to \langle e, f\rangle))$

$= (\langle a, b\rangle \to \langle e, f\rangle) \to ((\langle 1 - (1 - e).\mathrm{sg}(c - e),\ f.\mathrm{sg}(c - e),\ \mathrm{sg}(d - f)\rangle$
$\to \langle 1 - (1 - e).\mathrm{sg}(\max(a, c) - e), f.\mathrm{sg}(\max(a, c) - e).\mathrm{sg}(f - \min(b, d))\rangle)$

$= \langle 1 - (1 - e).\mathrm{sg}(a - e),\ f.\mathrm{sg}(a - e).\mathrm{sg}(f - b)\rangle$
$\to \langle 1 - (1 - e).\mathrm{sg}(\max(a, c) - e).\mathrm{sg}((1 - e).(\mathrm{sg}(\max(a, c) - b) - \mathrm{sg}(c - e))),$
$f.\mathrm{sg}(\max(a, c) - e).\mathrm{sg}(f - \min(b, d)).\mathrm{sg}((1 - e).(\mathrm{sg}(\max(a, c) - b) - \mathrm{sg}(c - e)))$
$.\mathrm{sg}(f.(\mathrm{sg}(\max(a, c) - e).\mathrm{sg}(f - \min(b, d)) - \mathrm{sg}(c - e).\mathrm{sg}(d - f)))\rangle$

$= \langle 1 - (1 - e).\mathrm{sg}(\max(a, c) - e).\mathrm{sg}((1 - e).(\mathrm{sg}(\max(a, c) - b) - \mathrm{sg}(c - e)))$
$.\mathrm{sg}((1 - e).(\mathrm{sg}(\max(a, c) - e).\mathrm{sg}((1 - e).(\mathrm{sg}(\max(a, c) - e) - \mathrm{sg}(c - e))) - \mathrm{sg}(a - e))),$
$f.\mathrm{sg}(\max(a, c) - e).\mathrm{sg}(f - \min(b, d)).\mathrm{sg}((1 - e).(\mathrm{sg}(\max(a, c) - b) - \mathrm{sg}(c - e)))$
$.\mathrm{sg}(f.((\mathrm{sg}(\max(a, c) - e).\mathrm{sg}(f - \min(b, d)) - \mathrm{sg}(c - e).\mathrm{sg}(d - f)))$
$.\mathrm{sg}(f.\mathrm{sg}(\max(a, c) - e).\mathrm{sg}(f - \min(b, d)).\mathrm{sg}((1 - e).(\mathrm{sg}(\max(a, c) - b) - \mathrm{sg}(c - e)))$
$.\mathrm{sg}(f.(\mathrm{sg}(\max(a, c) - e).\mathrm{sg}(f - \min(b, d)) - \mathrm{sg}(c - e).\mathrm{sg}(d - f))) - \mathrm{sg}(a - e).\mathrm{sg}(f - b)))\rangle$

if $a \ge c$ (because:
$\mathrm{sg}((1 - e).(\mathrm{sg}(\max(a, c) - e).\mathrm{sg}((1 - e).(\mathrm{sg}(\max(a, c) - e) - \mathrm{sg}(c - e))) - \mathrm{sg}(a - e)))$:
$= \mathrm{sg}((1 - e).\mathrm{sg}(a - e)\ .\ (\mathrm{sg}((1 - e).(\mathrm{sg}(a - e) - \mathrm{sg}(c - e)))) - 1) \le 0$)

$= \langle 1, 0\rangle$;

if $a < c$ (because:
$\mathrm{sg}((1 - e).(\mathrm{sg}(\max(a, c) - e).\mathrm{sg}((1 - e).(\mathrm{sg}(\max(a, c) - e) - \mathrm{sg}(c - e))) - \mathrm{sg}(a - e)))$:
$= \mathrm{sg}((1 - e).(\mathrm{sg}(c - e).\mathrm{sg}((1 - e).(\mathrm{sg}(c - e) - \mathrm{sg}(c - e))) - \mathrm{sg}(a - e)))$
$= \mathrm{sg}((1 - e).(0 - \mathrm{sg}(a - e))) = 0$)

$= \langle 1, 0\rangle$.

(i) $V(\neg\neg A \supset A) = V(\neg\neg A) \to V(A) = V(A) \to V(A) = \langle 1, 0\rangle$.

(j) $V((A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C)))$

$= (V(A) \to (V(B) \to V(C))) \to ((V(A) \to V(B)) \to (V(A) \to V(C)))$

$= (\langle a, b\rangle \to \langle 1 - (1 - e).\mathrm{sg}(c - e),\ f.\mathrm{sg}(c - e).\mathrm{sg}(f - d)\rangle) \to$
$(\langle 1 - (1 - c).\mathrm{sg}(a - c), d.\mathrm{sg}(a - c).\mathrm{sg}(d - b)\rangle \to \langle 1 - (1 - e).\mathrm{sg}(a - e),\ f.\mathrm{sg}(a - e).\mathrm{sg}(f - b)\rangle)$

$= \langle 1 - (1 - e).\mathrm{sg}(c - e).\mathrm{sg}(a - 1 + (1 - e).\mathrm{sg}(c - e)),$
$f.\mathrm{sg}(c - e).\mathrm{sg}(f - d).\mathrm{sg}(a - 1 + (1 - e).\mathrm{sg}(c - e)).\mathrm{sg}(f.\mathrm{sg}(c - e).\mathrm{sg}(f - d) - b)\rangle$
$\to \langle 1 - (1 - e).\mathrm{sg}(a - e).\mathrm{sg}((1 - e).\mathrm{sg}(a - e) - (1 - c).\mathrm{sg}(a - c)),$

$f.\text{sg}(a - e).\text{sg}(f - b).\text{sg}((1 - e), \text{sg}(a - e) - (1 - c).\text{sg}(a - c))$
$.\text{sg}(f.\text{sg}(a - e).\text{sg}(f - b) - d.\text{sg}(a - c).\text{sg}(d - b))\rangle$

$= \langle 1 - (1 - e).\text{sg}(a - e).\text{sg}((1 - e).\text{sg}(a - e) - (1 - c).\text{sg}(a - c))$
$.\text{sg}((1 - e).\text{sg}(a - e).\text{sg}((1 - e).\text{sg}(a - e) - (1 - c).\text{sg}(a - c))$
$- (1 - e).\text{sg}(c - e).\text{sg}(a - 1 + (1 - e).\text{sg}(c - e))),$
$f.\text{sg}(a - e).\text{sg}(f - b).\text{sg}((1 - e).\text{sg}(a - e) - (1 - c).\text{sg}(a - c))$
$.\text{sg}(f.\text{sg}(a - e).\text{sg}(f - b) - d.\text{sg}(a - c).\text{sg}(d - b)).\text{sg}((1 - e).\text{sg}(a - e)$
$.\text{sg}((1 - e).\text{sg}(a - e) - (1 - c).\text{sg}(a - c)) - (1 - e).\text{sg}(c - e).\text{sg}(a - 1 + (1 - e).\text{sg}(c - e)))$
$.\text{sg}(f.\text{sg}(a - e).\text{sg}(f - b).\text{sg}((1 - e).\text{sg}(a - e) - (1 - c).\text{sg}(a - c))$
$.\text{sg}(f.\text{sg}(a - e).\text{sg}(f - b) - d.\text{sg}(a - c).\text{sg}(d - b)) - f.\text{sg}(c - e).\text{sg}(f - d)$
$.\text{sg}(a - 1 + (1 - e).\text{sg}(c - e)).\text{sg}(f.\text{sg}(c - e).\text{sg}(f - d) - b))\rangle$

if $a \le e$:

$= \langle 1, 0\rangle$;

if $a > e$:

$= \langle 1 - (1 - e).\text{sg}((1 - e) - (1 - c).\text{sg}(a - c)).\text{sg}((1 - e).(\text{sg}((1 - e) - (1 - c).\text{sg}(a - c)))$
$- \text{sg}(c - e).\text{sg}(a - 1 + (1 - e).\text{sg}(c - e))),$
$f.\text{sg}(f - b).\text{sg}((1 - e) - (1 - c).\text{sg}(a - c)).\text{sg}(f.\text{sg}(f - b) - d.\text{sg}(a - c).\text{sg}(d - b))$
$.\text{sg}((1 - e).\text{sg}((1 - e) - (1 - c).\text{sg}(a - c)) - (1 - e).\text{sg}(c - e).\text{sg}(a - 1 + (1 - e).\text{sg}(c - e)))$
$.\text{sg}(f.\text{sg}(f - b).\text{sg}((1 - e) - (1 - c).\text{sg}(a - c)).\text{sg}(f.\text{sg}(f - b) - d.\text{sg}(a - c).\text{sg}(d - b))$
$- f.\text{sg}(c - e).\text{sg}(f - d).\text{sg}(a - 1 + (1 - e).\text{sg}(c - e)).\text{sg}(f.\text{sg}(c - e).\text{sg}(f - d) - b))\rangle$

if $a \le c$ (hence $c > e$ and in view of the equation for $x \ge 0$: $x.\text{sg}(x) = x$ we get:
$\text{sg}((1 - e) - \text{sg}(c - e).\text{sg}(a - 1 + (1 - e).\text{sg}(c - e)))$
$= \text{sg}(1 - e - \text{sg}(a - e)) = \text{sg}(1 - e - 1)) = \text{sg}(-e) = 0)$:

$= \langle 1, 0\rangle$;

if $a > c$ (for the same expression we get:
$\text{sg}((1 - e).\text{sg}((1 - e) - (1 - c)) - \text{sg}(c - e).\text{sg}(a - 1 + (1 - e).\text{sg}(c - e)))$
$= \text{sg}((1 - e).\text{sg}(c - e) - \text{sg}(c - e).\text{sg}(a - 1 + (1 - e).\text{sg}(c - e)))$

if $c \le e$:
$= \text{sg}(0) = 0$;

if $c > e$ (because $a > e$):
$= \text{sg}((1 - e) - \text{sg}(a - 1 + 1 - e)) = \text{sg}(1 - e - 1) = \text{sg}(-e) = 0)$:

$= \langle 1, 0\rangle$. □

In this way we find that the basic tautologies in the classical propositional calculus are tautologies in the IFPC. Only the classical tautology (see [3]):

(k) $(\neg A \supset \neg B) \supset ((\neg A \supset B) \supset A)$

is not valid.

2. (max-min)-variant of IFPC

Using the definitions for "&" and "$\vee$" above, we shall construct a new IFPC giving the following definition for "$\supset$":

$$V(p \supset q) = \langle \max(\nu(p), \mu(q)), \min(\mu(p), \nu(q))\rangle,$$

$$V(p \to q) = V(p \supset q).$$

For the needs of the discussion below, we shall define the notion intuitionistic fuzzy tautology (IFT) through:

"a is an lFT" <u>iff</u> "if $V(a) = \langle a, b\rangle$, then $a \geq b$".

For the so-defined operations, tautology and evaluation, using the notations above, we shall prove:

Theorem 3: If A, B and C are propositional forms, then (a)-(j) from Theorem 2 and the classical tautology (k) are IFTs.

Proof:

(a) $V(A \supset A) = V(A) \to V(A)$

$= \langle \max(a, a), \min(a, a)\rangle,$

and $\max(a, a) \geq \min(a, a)$.

(b) $V(A \supset (B \supset A)) = V(A) \to (V(B) \to V(A))$

$= \langle a, b\rangle \to \langle \max(a, d), \min(b, c)\rangle$

$= \langle \max(a, b, d), \min(a, b, c)\rangle,$

and $\max(a, b, d) \geq a \geq \min(a, b, c)$.

(c)-(f) and (i) are proved analogically.

(g) $V(A \supset (B \supset (A \,\&\, B)))$

$= \langle a, b\rangle \to (\langle c, d\rangle \to \langle \min(a, c), \max(b, d)\rangle)$

$= \langle a, b\rangle \to \langle \max(d, \min(a, c)), \min(c, \max(b, d))\rangle$

$= \langle \max(b, d, \min(a, c)), \min(a, c, \max(b, d))\rangle,$

and $\max(b, d, \min(a, c)) \geq \max(b, d) \geq \min(a, c, \max(b, d))$.

(h) $V((A \supset C) \supset ((B \supset C) \supset ((A \vee B) \supset C)))$

$= \langle \max(b, e), \min(a, f)\rangle \to (\langle \max(d, e), \min(c, f)\rangle \to (\langle \max(a, c), \min(b, d)\rangle \to \langle e, f\rangle))$

$= \langle \max(b, e), \min(a, f)\rangle \to (\langle \max(d, e), \min(c, f)\rangle$
$\to (\langle \max(e, \min(b, d)), \min(f, \max(a, c))\rangle)$

$= \langle \max(b, e), \min(a, f)\rangle \to \langle \max(\min(c, f), e, \min(b, d)), \min(\max(d, e), f, \max(a, c))\rangle$

$= \langle \max(\min(a, f), \min(c, f), e, \min(b, d)), \min(\max(b, e), \max(d, e), f, \max(a, c))\rangle,$

and $\max(\min(a, f), \min(c, f), e, \min(b, d))$

$\geq \max(\min(a, f), \min(c, f)) = \min(f, \max(a, c))$

$$\geq \min(\max(b, e), \max(d, e), f, \max(a, c)).$$

(j) $V((A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C)))$

$= (\langle a, b\rangle \to \langle \max(d, e), \min(c, f)\rangle \to (\langle \max(b, c), \min(a, d)\rangle \to \langle \max(b, e), \min(a, f)\rangle)$

$= \langle \max(b, d, e), \min(a, c, f)\rangle \to \langle \max(b, e, \min(a, d)), \min(a, d, \max(b, c))\rangle$

$= \langle \max(b, e, \min(a, d), \min(a, c, f)), \min(a, d), \max(b, c), \max(b, d, e))\rangle,$

and $\max(b, e, \min(a, d), \min(a, c, f))$

$\geq \max(b, e, \min(a, d)) = \min(a, \max(b, d, e))$

$\geq \min(a, d, \max(b, c), \max(b, d, e)).$

(k) $V((\neg A \supset \neg B) \supset ((\neg A \supset B) \supset A))$

$= (\langle b, a\rangle \to \langle d, c\rangle) \to (\langle \max(a, c), \min(b, d)\rangle \to \langle a, b\rangle)$

$= \langle \max(a, d), \min(b, c)\rangle \to \langle \max(a, \min(b, d)), \min(b, \max(a, c))\rangle$

$= \langle \max(\min(b, c), a, \min(b, d)), \min(b, \max(a, c), \max(a, d))\rangle,$

and $\max(\min(b, c), a, \min(b, d))$

$\geq \max(a, \min(b, c)) \geq \min(b, \max(a, c))$

$\geq \min(b, \max(a, c), \max(a, d)).$ □

With this choice of operations, tautology, and evaluation it turns out that the Modus Ponens is not valid, on another hand a well-known equality in the classical logical is valid:

$$\langle a, b\rangle \supset \langle 0, 1\rangle = \langle b, a\rangle.$$

The IFPC (sg- or (max-min)-version) can be used as a basis for construction of intuitionistic fuzzy expert systems and intuitionistic fuzzy PROLOG (c.f. [4]).

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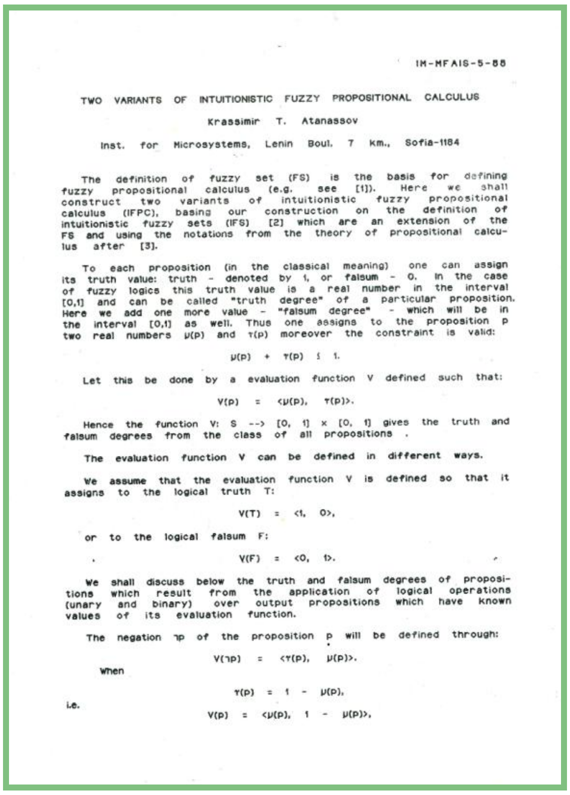

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TWO VARIANTS OF INTUITIONISTIC FUZZY PROPOSITIONAL CALCULUS

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The definition of fuzzy set (FS) is the basis for defining fuzzy propositional calculus (e.g. see [1]). Here we shall construct two variants of intuitionistic fuzzy propositional calculus (IFPC), basing our construction on the definition of intuitionistic fuzzy sets (IFS) [2] which are an extension of the FS and using the notations from the theory of propositional calculus after [3].

To each proposition (in the classical meaning) one can assign its truth value: truth - denoted by 1, or falsum - 0. In the case of fuzzy logics this truth value is a real number in the interval [0,1] and can be called "truth degree" of a particular proposition. Here we add one more value - "falsum degree" - which will be in the interval [0,1] as well. Thus one assigns to the proposition p two real numbers $\mu(p)$ and $\gamma(p)$ moreover the constraint is valid:

$$\mu(p) + \gamma(p) \leq 1.$$

Let this be done by a evaluation function V defined such that:

$$V(p) = \langle \mu(p), \gamma(p) \rangle.$$

Hence the function $V: S \dashrightarrow [0, 1] \times [0, 1]$ gives the truth and falsum degrees from the class of all propositions .

The evaluation function V can be defined in different ways.

We assume that the evaluation function V is defined so that it assigns to the logical truth T:

$$V(T) = \langle 1, 0 \rangle,$$

or to the logical falsum F:

$$V(F) = \langle 0, 1 \rangle.$$

We shall discuss below the truth and falsum degrees of propositions which result from the application of logical operations (unary and binary) over output propositions which have known values of its evaluation function.

The negation $\neg p$ of the proposition p will be defined through:

$$V(\neg p) = \langle \gamma(p), \mu(p) \rangle.$$

When

$$\gamma(p) = 1 - \mu(p),$$

i.e.

$$V(p) = \langle \mu(p), 1 - \mu(p) \rangle,$$

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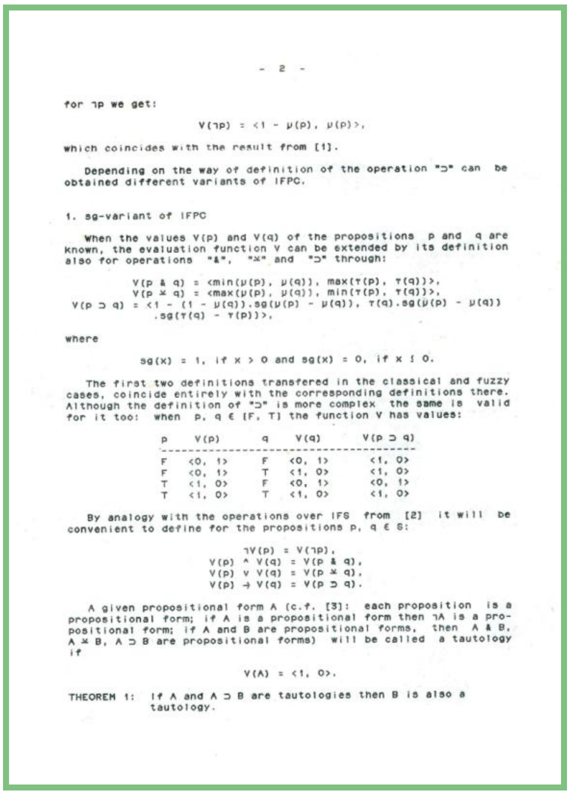

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for $\neg p$ we get:

$$V(\neg p) = \langle 1 - \mu(p), \mu(p) \rangle,$$

which coincides with the result from [1].

Depending on the way of definition of the operation "$\supset$" can be obtained different variants of IFPC.

1\. sg-variant of IFPC

When the values V(p) and V(q) of the propositions p and q are known, the evaluation function V can be extended by its definition also for operations "&", "$\veebar$" and "$\supset$" through:

$$V(p \& q) = \langle \min(\mu(p), \mu(q)), \max(\gamma(p), \gamma(q)) \rangle,$$
$$V(p \veebar q) = \langle \max(\mu(p), \mu(q)), \min(\gamma(p), \gamma(q)) \rangle,$$
$$V(p \supset q) = \langle 1 - (1 - \mu(q)).sg(\mu(p) - \mu(q)), \gamma(q).sg(\mu(p) - \mu(q)).sg(\gamma(q) - \gamma(p)) \rangle,$$

where

$$sg(x) = 1, \text{ if } x > 0 \text{ and } sg(x) = 0, \text{ if } x \leq 0.$$

The first two definitions transfered in the classical and fuzzy cases, coincide entirely with the corresponding definitions there. Although the definition of "$\supset$" is more complex the same is valid for it too: when $p, q \in \{F, T\}$ the function V has values:

p	V(p)	q	V(q)	V(p ⊃ q)
F	<0, 1>	F	<0, 1>	<1, 0>
F	<0, 1>	T	<1, 0>	<1, 0>
T	<1, 0>	F	<0, 1>	<0, 1>
T	<1, 0>	T	<1, 0>	<1, 0>

By analogy with the operations over IFS from [2] it will be convenient to define for the propositions $p, q \in S$:

$$\neg V(p) = V(\neg p),$$
$$V(p) \wedge V(q) = V(p \& q),$$
$$V(p) \vee V(q) = V(p \veebar q),$$
$$V(p) \rightarrow V(q) = V(p \supset q).$$

A given propositional form A (c.f. [3]: each proposition is a propositional form; if A is a propositional form then $\neg A$ is a propositional form; if A and B are propositional forms, then A & B, $A \veebar B$, $A \supset B$ are propositional forms) will be called a tautology if

$$V(A) = \langle 1, 0 \rangle.$$

THEOREM 1: If A and $A \supset B$ are tautologies then B is also a tautology.

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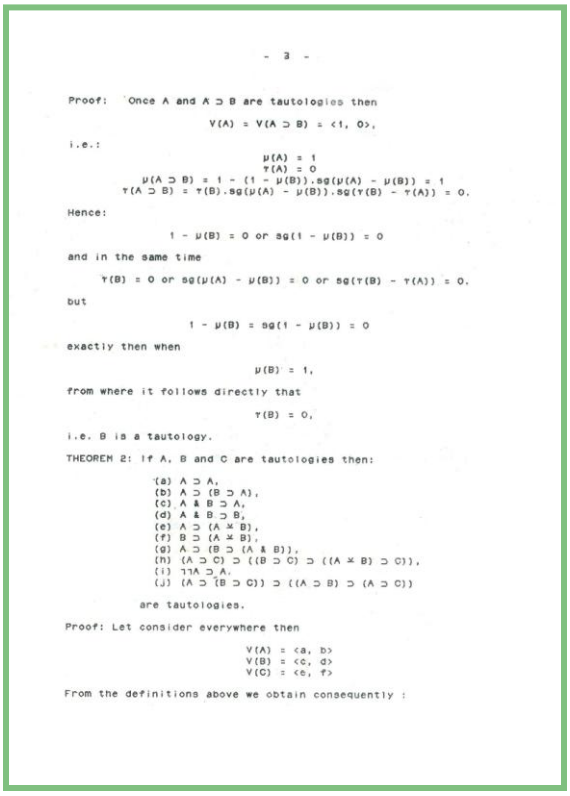

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Proof: Once A and $A \supset B$ are tautologies then

$$V(A) = V(A \supset B) = \langle 1, 0 \rangle,$$

i.e.:

$$\mu(A) = 1$$
$$\gamma(A) = 0$$
$$\mu(A \supset B) = 1 - (1 - \mu(B)).sg(\mu(A) - \mu(B)) = 1$$
$$\gamma(A \supset B) = \gamma(B).sg(\mu(A) - \mu(B)).sg(\gamma(B) - \gamma(A)) = 0.$$

Hence:

$$1 - \mu(B) = 0 \text{ or } sg(1 - \mu(B)) = 0$$

and in the same time

$$\gamma(B) = 0 \text{ or } sg(\mu(A) - \mu(B)) = 0 \text{ or } sg(\gamma(B) - \gamma(A)) = 0.$$

but

$$1 - \mu(B) = sg(1 - \mu(B)) = 0$$

exactly then when

$$\mu(B) = 1,$$

from where it follows directly that

$$\gamma(B) = 0,$$

i.e. B is a tautology.

THEOREM 2: If A, B and C are tautologies then:

(a) $A \supset A$,
(b) $A \supset (B \supset A)$,
(c) $A \& B \supset A$,
(d) $A \& B \supset B$,
(e) $A \supset (A \veebar B)$,
(f) $B \supset (A \veebar B)$,
(g) $A \supset (B \supset (A \& B))$,
(h) $(A \supset C) \supset ((B \supset C) \supset ((A \veebar B) \supset C))$,
(i) $\neg\neg A \supset A$,
(j) $(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$

are tautologies.

Proof: Let consider everywhere then

$$V(A) = \langle a, b \rangle$$
$$V(B) = \langle c, d \rangle$$
$$V(C) = \langle e, f \rangle$$

From the definitions above we obtain consequently :

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(a) $V(A \supset A) = V(A) \rightarrow V(A)$
$= \langle 1-(1-a).sg(a-a), b.sg(a-a).sg(b-b) \rangle$
$= \langle 1, 0 \rangle.$

(b) $V(A \supset (B \supset A)) = V(A) \rightarrow (V(B) \rightarrow V(A))$
$= \langle a, b \rangle \rightarrow \langle 1-(1-a).sg(c-a), b.sg(c-a).sg(b-d) \rangle$
$= \langle 1-(1-(1-(1-a).sg(c-a))).sg(a-1+(1-a).sg(c-a)),$
$b.sg(c-a).sg(b-d).sg(a-1+(1-a).sg(c-a)).sg(b.sg(c-a).sg(b-d)-b) \rangle$

if $a \geq c$:
$= \langle 1, 0 \rangle;$

if $a < c$:
$= \langle 1-(1-a).sg(a-1+(1-a).1), b.sg(b-d).sg(a-1+(1-a).1)$
$.sg(b.sg(b-d)-b) \rangle$
$= \langle 1, 0 \rangle.$

(c) $V(A \& B \supset A) = \langle \min(a,c), \max(b,d) \rangle \rightarrow \langle a, b \rangle$
$= \langle 1-(1-a).sg(\min(a,c)-a), b.sg(\min(a,c)-a).sg(b-\max(b,d)) \rangle$
$= \langle 1, 0 \rangle.$

(d) is proved analogically.

(e) $V(A \supset (A \veebar B)) = \langle a, b \rangle \rightarrow \langle \max(a,c), \min(b,d) \rangle$
$= \langle 1-(1-\max(a,c)).sg(a-\max(a,c)), \min(b,d).sg(a-\max(a,c))$
$.sg(\min(b,d)-b) \rangle$
$= \langle 1, 0 \rangle$

(f) is proved analogically.

(g) $V(A \supset (B \supset (A \& B)))$
$= \langle a, b \rangle \rightarrow (\langle c, d \rangle \rightarrow \langle \min(a,c), \max(b,d) \rangle)$
$= \langle a, b \rangle \rightarrow \langle 1-(1-\min(a,c)).sg(c-\min(a,c)), \max(b,d).sg(c-\min(a,c)$
$.sg(\max(b,d)-d) \rangle$
$= \langle 1-(1-\min(a,c)).sg(c-\min(a,c)).sg(a-1+(1-\min(a,c)).sg(c$
$+\min(a,c)), \max(b,d).sg(c-\min(a,c)).sg(\max(b,d)-d).sg(a-1$
$+\min(a,c)).sg(c-\min(a,c))).sg(\max(b,d).sg(c-\min(a,c))$
$.sg(\max(b,d)-d)-b) \rangle$

(from:
$a-1+(1-\min(a,c)).sg(c-\min(a,c)) \leq a-1+(1-a).1 = 0)$

$= \langle 1, 0 \rangle.$

(h) $V((A \supset C) \supset ((B \supset C) \supset ((A \veebar B) \supset C)))$
$= (\langle a, b \rangle \rightarrow \langle e, f \rangle) \rightarrow ((\langle c, d \rangle \rightarrow \langle e, f \rangle) \rightarrow$
$(\langle \max(a,c), \min(b,d) \rangle \rightarrow \langle e,f \rangle))$
$= (\langle a, b \rangle \rightarrow \langle e, f \rangle) \rightarrow (\langle 1-(1-e).sg(c-e), f.sg(c-e).sg(d-f) \rangle \rightarrow$
$\langle 1-(1-e).sg(\max(a,c)-e), f.sg(\max(a,c)-e).sg(f-\min(b,d)) \rangle)$
$= \langle 1-(1-e).sg(a-e), f.sg(a-e).sg(f-b) \rangle \rightarrow \langle 1-(1-e).sg(\max(a,c)-e)$
$.sg((1-e).(sg(\max(a,c)-b)-sg(c-e))), f.sg(\max(a,c)-e)$
$.sg(f-\min(b,d)).sg((1-e).(sg(\max(a,c)-b)-sg(c-e)))$
$.sg(f.(sg(\max(a,c)-e).sg(f-\min(b,d))-sg(c-e).sg(d-f))) \rangle$
$= \langle 1-(1-e).sg(\max(a,c)-e).sg((1-e).(sg(\max(a,c)-b)-sg(c-e)))$
$.sg((1-e).(sg(\max(a,c)-e).sg((1-e).(sg(\max(a,c)-e)-sg(c-e)))$
$-sg(a-e))), f.sg(\max(a,c)-e).sg(f-\min(b,d)).sg((1-e).$

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$$\begin{aligned}&(sg(\max(a,c)-b)-sg(c-e))).sg(f.((sg(\max(a,c)-e).sg(f-\min(b,d))\\&-sg(c-e).sg(d-f))).sg(f.sg(\max(a,c)-e).sg(f-\min(b,d)).sg((1-e)\\&.(sg(\max(a,c)-b)-sg(c-e))).sg(f.(sg(\max(a,c)-e).sg(f-\min(b,d))\\&-sg(c-e).sg(d-f)))-sg(a-e).sg(f-b)))\rangle\end{aligned}$$

if $a \geq c$ (because:

$$\begin{aligned}&sg((1-e).(sg(\max(a,c)-e).sg((1-e).(sg(\max(a,c)-e)-sg(c-e)))\\&-sg(a-e))):\\&= sg((1-e).sg(a-e).(sg((1-e).(sg(a-e)-sg(c-e))))-1) \leq 0)\\&= \langle 1, 0\rangle;\end{aligned}$$

if $a < c$ (because:

$$\begin{aligned}&sg((1-e).(sg(\max(a,c)-e).sg((1-e).(sg(\max(a,c)-e)-sg(c-e)))\\&-sg(a-e))):\\&= sg((1-e).(sg(c-e).sg((1-e).(sg(c-e)-sg(c-e)))-sg(a-e)))\\&= sg((1-e).(0-sg(a-e))) = 0)\\&= \langle 1, 0\rangle.\end{aligned}$$

(i) $V(\neg\neg A \supset A) = V(\neg\neg A) \to V(A) = V(A) \to V(A) = \langle 1, 0\rangle$.

(j) $$\begin{aligned}&V((A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C)))\\&= (V(A) \to (V(B) \to V(C))) \to ((V(A) \to V(B)) \to (V(A) \to V(C)))\\&= (\langle a, b\rangle \to \langle 1-(1-e).sg(c-e), f.sg(c-e).sg(f-d)\rangle) \to\\&\quad(\langle 1-(1-c).sg(a-c), d.sg(a-c).sg(d-b)\rangle \to \langle 1-(1-e).sg(a-e),\\&\quad f.sg(a-e).sg(f-b)\rangle)\\&= \langle 1-(1-e).sg(c-e).sg(a-1+(1-e).sg(c-e)), f.sg(c-e).sg(f-d)\\&\quad.sg(a-1+(1-e).sg(c-e)).sg(f.sg(c-e).sg(f-d)-b)\rangle \to\\&\quad\langle 1-(1-e).sg(a-e).sg((1-e).sg(a-e)-(1-c).sg(a-c)),\\&\quad f.sg(a-e).sg(f-b).sg((1-e).sg(a-e)-(1-c).sg(a-c))\\&\quad.sg(f.sg(a-e).sg(f-b)-d.sg(a-c).sg(d-b))\rangle\\&= \langle 1-(1-e).sg(a-e).sg((1-e).sg(a-e)-(1-c).sg(a-c))\\&\quad.sg((1-e).sg(a-e).sg((1-e).sg(a-e)-(1-c).sg(a-c))\\&\quad-(1-e).sg(c-e).sg(a-1+(1-e).sg(c-e))), f.sg(a-e).sg(f-b)\\&\quad.sg((1-e).sg(a-e)-(1-c).sg(a-c)).sg(f.sg(a-e).sg(f-b)-d.sg(a-c)\\&\quad.sg(d-b)).sg((1-e).sg(a-e).sg((1-e).sg(a-e)-(1-c).sg(a-c))\\&\quad-(1-e).sg(c-e).sg(a-1+(1-e).sg(c-e))).sg(f.sg(a-e).sg(f-b)\\&\quad.sg((1-e).sg(a-e)-(1-c).sg(a-c)).sg(f.sg(a-e).sg(f-b)-d.sg(a-c)\\&\quad.sg(d-b))-f.sg(c-e).sg(f-d).sg(a-1+(1-e).sg(c-e)).sg(f.sg(c-e)\\&\quad.sg(f-d)-b))\rangle\end{aligned}$$

if $a \leq e$:

$= \langle 1, 0\rangle$;

if $a > e$:

$$\begin{aligned}&= \langle 1-(1-e).sg((1-e)-(1-c).sg(a-c)).sg((1-e).(sg((1-e)-\\&\quad(1-c).sg(a-c)))-sg(c-e).sg(a-1+(1-e).sg(c-e))), f.sg(f-b)\\&\quad.sg((1-e)-(1-c).sg(a-c)).sg(f.sg(f-b)-d.sg(a-c).sg(d-b)).\\&\quad sg((1-e).sg((1-e)-(1-c).sg(a-c))-(1-e).sg(c-e).sg(a-1+(1-e)\\&\quad.sg(c-e))).sg(f.sg(f-b).sg((1-e)-(1-c).sg(a-c)).sg(f.sg(f-b)-\\&\quad d.sg(a-c).sg(d-b))-f.sg(c-e).sg(f-d).sg(a-1+(1-e).sg(c-e))\\&\quad.sg(f.sg(c-e).sg(f-d)-b))\rangle\end{aligned}$$

if $a \leq c$ (hence $c > e$ and in view of the equation for $x \geq 0$:

$$x.sg(x) = x$$

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we get:

$$\begin{aligned}&sg((1-e)-sg(c-e).sg(a-1+(1-e).sg(c-e)))\\&= sg(1-e-sg(a-e)) = sg(1 - e - 1)) = sg(-e) = 0):\end{aligned}$$

$= \langle 1, 0\rangle$;

if $a > c$ (for the same expression we get:

$$\begin{aligned}&sg((1-e).sg((1-e)-(1-c))-sg(c-e).sg(a-1+(1-e).sg(c-e)))\\&= sg((1-e).sg(c-e)-sg(c-e).sg(a-1+(1-e).sg(c-e)))\end{aligned}$$

if $c \leq e$:
$= sg(0) = 0$;

if $c > e$ (because $a > e$):
$= sg((1-e)-sg(a-1+1-e)) = sg(1 - e - 1) = sg(-e) = 0)$:

$= \langle 1, 0\rangle$.

In this way we find that the basic tautologies in the classical propositional calculus are tautologies in the IFPC. Only the classical tautology (see [3]):

$$(K) \quad (\neg A \supset \neg B) \supset ((\neg A \supset B) \supset A)$$

is not valid.

2. (max-min)-variant of IFPC

Using the definitions for "&" and "∨" above we shall construct a new IFPC giving the following definition for "⊃":

$$V(p \supset q) = \langle \max(\gamma(p), \mu(q)), \min(\mu(p), \gamma(q))\rangle$$

$$V(p) \to V(q) = V(p \supset q).$$

For the needs of the discussion below we shall define the notion intuitionistic fuzzy tautology (IFT) through:

"A is an IFT" iff "if $V(A) = \langle a, b\rangle$, then $a \geq b$".

For the so-defined operations, tautology and evaluation, using the notations above, we shall prove

THEOREM 3: If A, B and C are propositional forms, then (a)-(j) from Theorem 2 and the classical tautology (K) are IFTs.

Proof:

(a) $V(A \supset A) = V(A) \to V(B)$
$= \langle \max(a, b), \min(a, b)\rangle$,

and
$\max(a, b) \geq \min(a, b)$.

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(b) $$\begin{aligned}&V(A \supset (B \supset A)) = V(A) \to (V(B) \to V(A))\\&= \langle a, b\rangle \to \langle \max(a, d), \min(b, c)\rangle\\&= \langle \max(a, b, d), \min(a, b, c)\rangle\end{aligned}$$

and
$\max(a, b, d) \geq a \geq \min(a, b, c)$.

(c) - (f) and (i) are proved analogically.

(g) $$\begin{aligned}&V(A \supset (B \supset (A \,\&\, B)))\\&= \langle a, b\rangle \to (\langle c, d\rangle \to \langle \min(a, c), \max(b, d)\rangle)\\&= \langle a, b\rangle \to \langle \max(d, \min(a,c)), \min(c, \max(b,d))\rangle\\&= \langle \max(b, d, \min(a,c)), \min(a, c, \max(b,d))\rangle\end{aligned}$$

and
$\max(b, d, \min(a,c)) \geq \max(b, d) \geq \min(a, c, \max(b,d))$.

(h) $$\begin{aligned}&V((A \supset C) \supset ((B \supset C) \supset ((A \vee B) \supset C)))\\&= \langle \max(b, e), \min(a, f)\rangle \to (\langle \max(d, e), \min(c, f)\rangle \to\\&\quad(\langle \max(a, c), \min(b, d)\rangle \to \langle e, f\rangle))\\&= \langle \max(b, e), \min(a, f)\rangle \to (\langle \max(d, e), \min(c, f)\rangle \to\\&\quad\langle \max(e, \min(b, d)), \min(f, \max(a, c))\rangle)\\&= \langle \max(b, e), \min(a, f)\rangle \to \langle \max(\min(c, f), e, \min(b, d)),\\&\quad\min(\max(d, e), f, \max(a, c))\rangle\\&= \langle \max(\min(a, f), \min(c, f), e, \min(b, d)), \min(\max(b, e),\\&\quad\max(d, e), f, \max(a, c))\rangle\end{aligned}$$

and
$$\begin{aligned}&\max(\min(a, f), \min(c, f), e, \min(b, d))\\&\geq \max(\min(a, f), \min(c, f)) = \min(f, \max(a, c))\\&\geq \min(\max(b, e), \max(d, e), f, \max(a, c)).\end{aligned}$$

(j) $$\begin{aligned}&V((A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C)))\\&= (\langle a, b\rangle \to \langle \max(d, e), \min(c, f)\rangle) \to (\langle \max(b, c), \min(a, d)\rangle\\&\quad\to \langle \max(b, e), \min(a, f)\rangle)\\&= \langle \max(b, d, e), \min(a, c, f)\rangle \to \langle \max(b, e, \min(a, d)),\\&\quad\min(a, d, \max(b, c))\rangle\\&= \langle \max(b, e, \min(a, d), \min(a, c, f)), \min(a, d, \max(b, c),\\&\quad\max(b, d, e))\rangle\end{aligned}$$

and
$$\begin{aligned}&\max(b, e, \min(a, d), \min(a, c, f))\\&\geq \max(b, e, \min(a, d))\\&\geq \min(a, \max(b, d, e))\\&\geq \min(a, d, \max(b, c), \max(b, d, e)).\end{aligned}$$

(K) $$\begin{aligned}&V((\neg A \supset \neg B) \supset ((\neg A \supset B) \supset A))\\&= (\langle b, a\rangle \to \langle d, c\rangle) \to (\langle \max(a, c), \min(b, d)\rangle \to \langle a, b\rangle)\\&= \langle \max(a, d), \min(b, c)\rangle \to \langle \max(a, \min(b, d)), \min(b,\\&\quad\max(a, c))\rangle\\&= \langle \max(\min(b, c), a, \min(b, d)), \min(b, \max(a, c), \max(a, d))\rangle\end{aligned}$$

and
$$\begin{aligned}&\max(\min(b, c), a, \min(b, d))\\&\geq \max(a, \min(b, c))\\&\geq \min(b, \max(a, c))\\&\geq \min(b, \max(a, c), \max(a, d)).\end{aligned}$$

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With this choice of operations, tautology, and evaluation it turns out that the Modus Ponens is not valid. On another hand a well known equality in the classical logical is valid:

$$\langle a, b\rangle \supset \langle 0, 1\rangle = \langle b, a\rangle.$$

The IFPC (sg- or (max-min)-version) can be used as a basis for construction of intuitionistic fuzzy expert systems and intuitionistic fuzzy PROLOG (c.f. [4]).

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