Geometrical Interpretation of the Elements of the Intuitionistic Fuzzy Objects

Krassimir T. Atanassov*

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The concepts intuitionistic fuzzy set (IFS), intuitionistic fuzzy propositional calculus (IFPC), intuitionistic fuzzy modal logic (IFML) has been investigated yet only as logical and algebraic objects (c.f. [1-6]). Here we shall give two geometrical interpretations of the elements of IFSs, IFPC and IFML.

Let the universe E be given and let the figure F in the Euclidian plane with Cartesian coordinates be given (see Fig. 1):

$$F = \{p \mid (p = \langle a, b\rangle) \,\&\, (a, b \geq 0) \,\&\, (a + b \leq 1)\}.$$

Let $A \subset E$ be a fixed set. Then we can construct a function f_A from E to F, that if $x \in E$, then

$$f_A(x) = p = \langle a, b\rangle \in F,$$
$$0 \leq a + b \leq 1.$$

These coordinates are such that:

$$a = \mu_A(x),$$
$$b = \nu_A(x).$$

Therefore the function f_A is a surjection, because every two different elements of F can be images of elements of E about f_A. The opposite is not valid. For example, if $x, y \in E$ but $x \neq y$ and $\mu_A(x) = \mu_A(y)$, $\nu_A(x) = \nu_A(y)$ about the set $A \subset E$, then $f_A(x) = f_A(y)$.

Obviously, f_A will assign the element with coordinates $\langle 1, 0\rangle$ to the element $x \in E$ for which:

$$\mu_A(x) = 1,$$
$$\nu_A(x) = 0,$$

and f_A will assign the element with coordinates $\langle 0, 1\rangle$ to the element $x \in E$ for which:

$$\mu_A(x) = 0,$$
$$\nu_A(x) = 1.$$

* Current affiliation: Bioinformatics and Mathematical Modelling Department
Institute of Biophysics and Biomedical Engineering, Bulgarian Academy of Sciences
105 Acad. G. Bonchev Str., Sofia 1113, Bulgaria, E-mail: krat@bas.bg

The geometrical interpretation of the operations and operators over the elements of the IFS, IFPC and IFML is important for the fact, that they become visualizable.

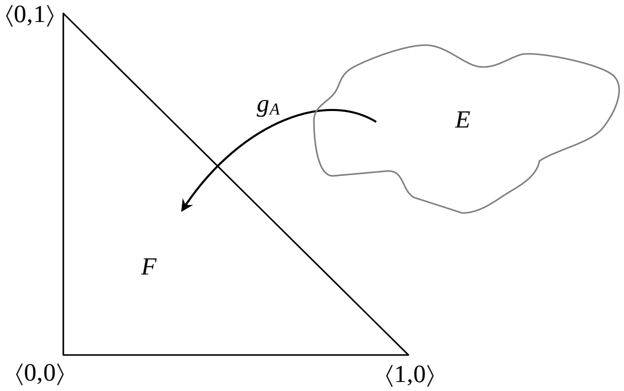


Fig. 1

If $x, y \in E$, then to $x \& y$ f_A will assign a point $f_A(x \& y) \in F$ with coordinates $\langle \min(\mu_A(x), \mu_A(y)), \max(\nu_A(x), \nu_A(y))\rangle$. There exist three geometrical cases (see Fig. 2 a-c) – one general case (Fig. 2a) and two particular cases (Fig. 2b and 2c).

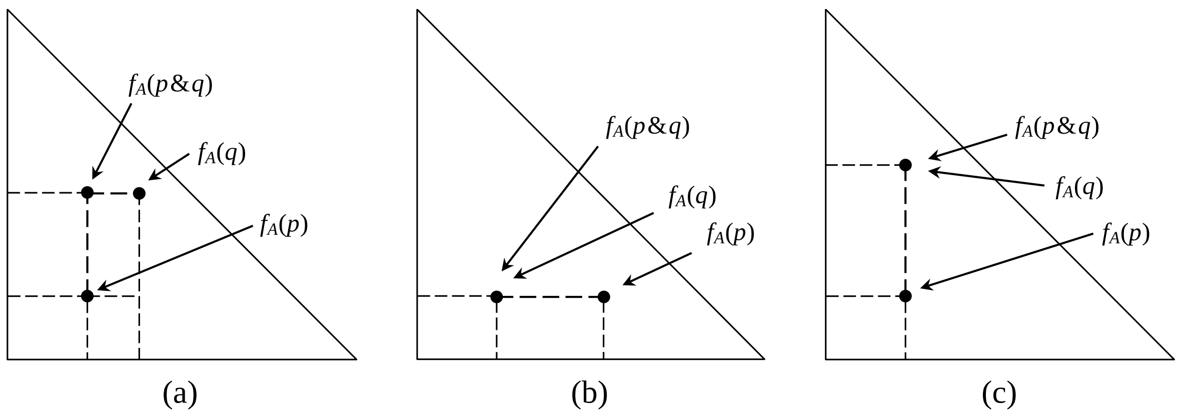


Fig. 2

If $x, y \in E$, then to $x \vee y$ f_A will assign a point $f_A(x \vee y) \in F$ with coordinates $\langle \max(\mu_A(x), \mu_A(y)), \min(\nu_A(x), \nu_A(y))\rangle$. There exist also three geometrical cases as above – one general case (Fig. 3) and two particular cases.

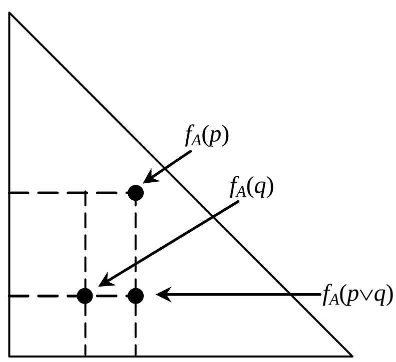


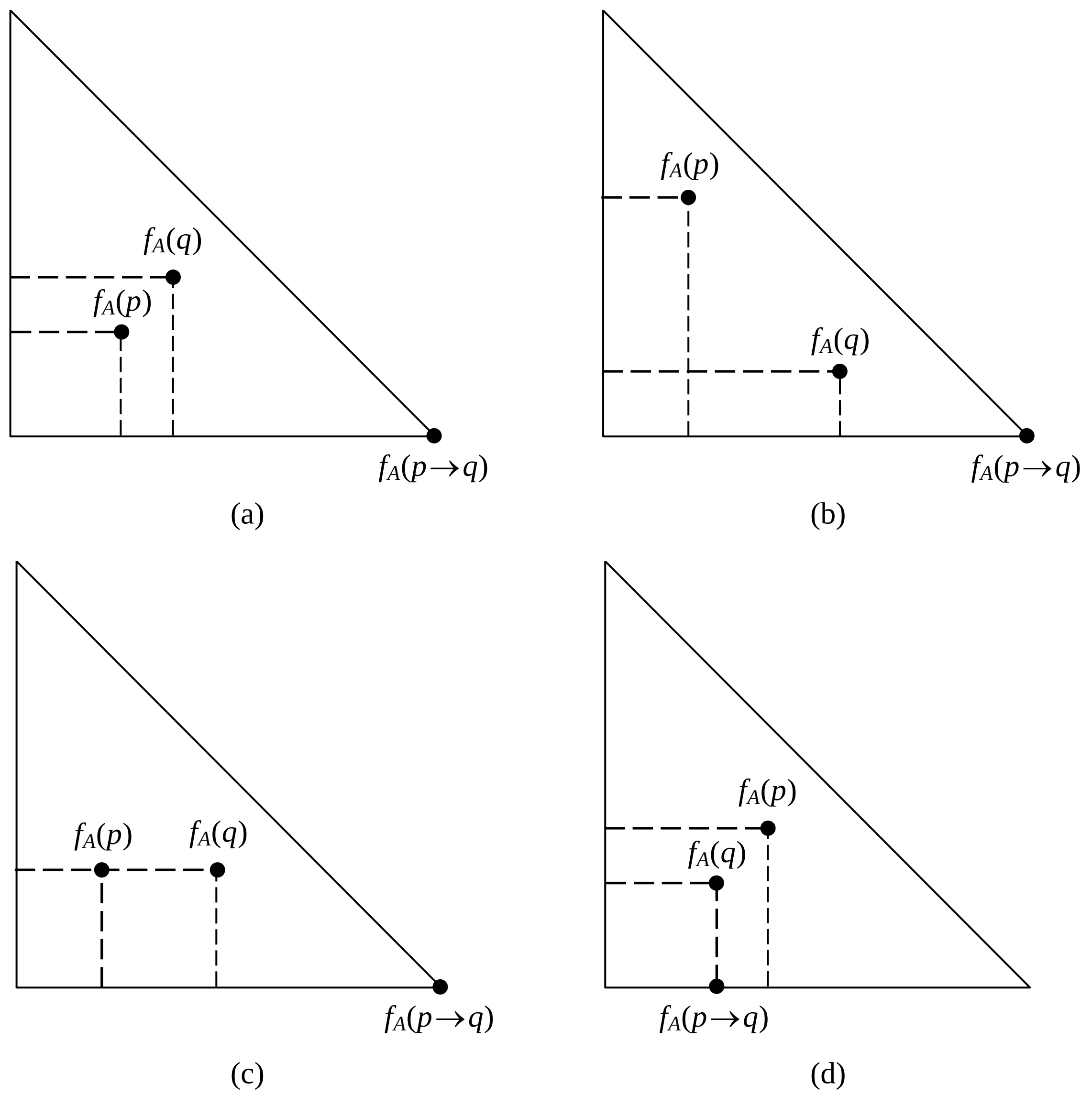
Fig. 3

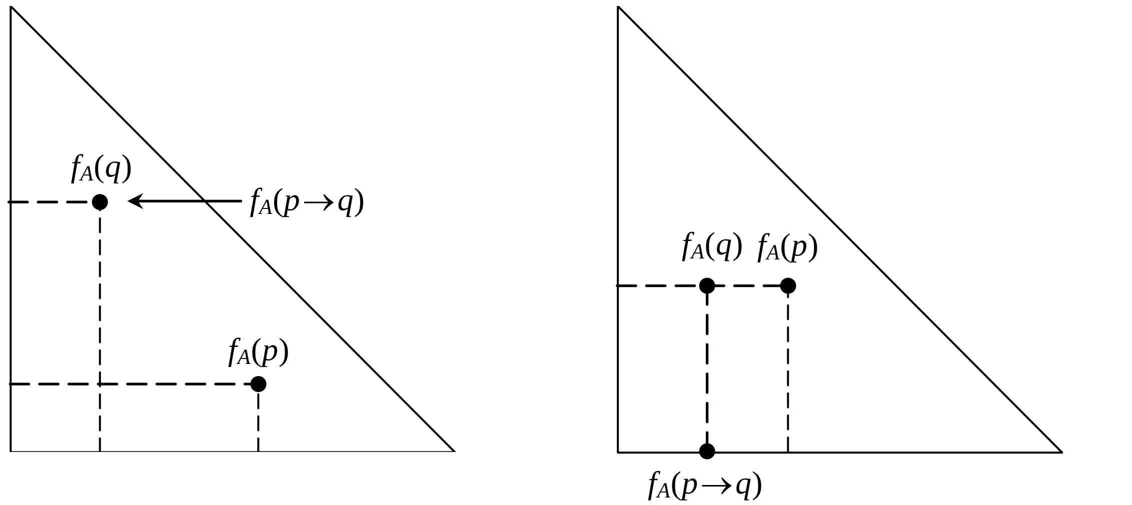
More complex is the case of the element $x \to y$ (c.f. [4]) because for it there exist at least two different versions of a definition.

In the first case, f_A assigns to the element $x \to y \in E$ the element $f_A(x \to y) \in F$ with coordinates $\langle 1 - (1 - \mu_A(y)).sg(\mu_A(x)) - \mu_A(y)), \nu_A(y).sg(\mu_A(x) - \mu_A(y)).sg(\nu_A(y) - \nu_A(x))\rangle$ (Fig. 4a).

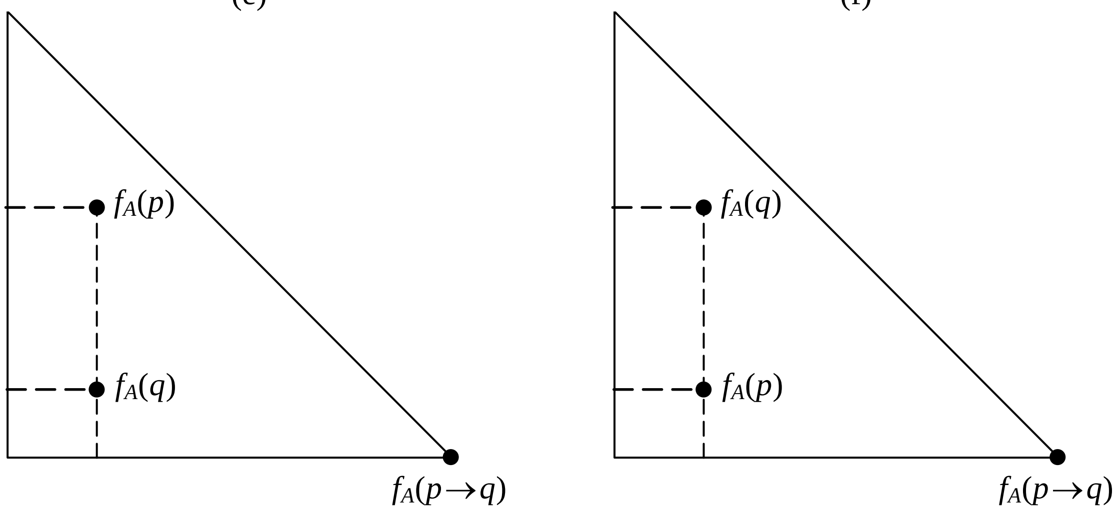
In contrast to the case for "&", here the locations of the points $f_A(x)$ and $f_A(y)$ are important for the form of the location of the point $f_A(x \to y)$. The locations in Fig. 4a and 4b are general and in Fig. 4c – a special case; in Fig. 4d and 4e are general and in Fig. 4f – a special case; in Fig. 4g and 4h are particular cases of these from Fig. 4a and 4b.

In the second case the point $f_A(x \to y)$ has coordinates $\langle \max(\mu_A(y), \nu_A(x)), \min(\nu_A(x), \mu_A(y))\rangle$ and the locations of the points $f_A(x)$ and $f_A(y)$ are important for the form of the location of the point $f_A(x \to y)$ also – see Fig. 5a-h.




fA(q)
fA(p→q)
fA(p)
fA(q) fA(p)
fA(p→q)

(e) (f)


fA(p)
fA(q)
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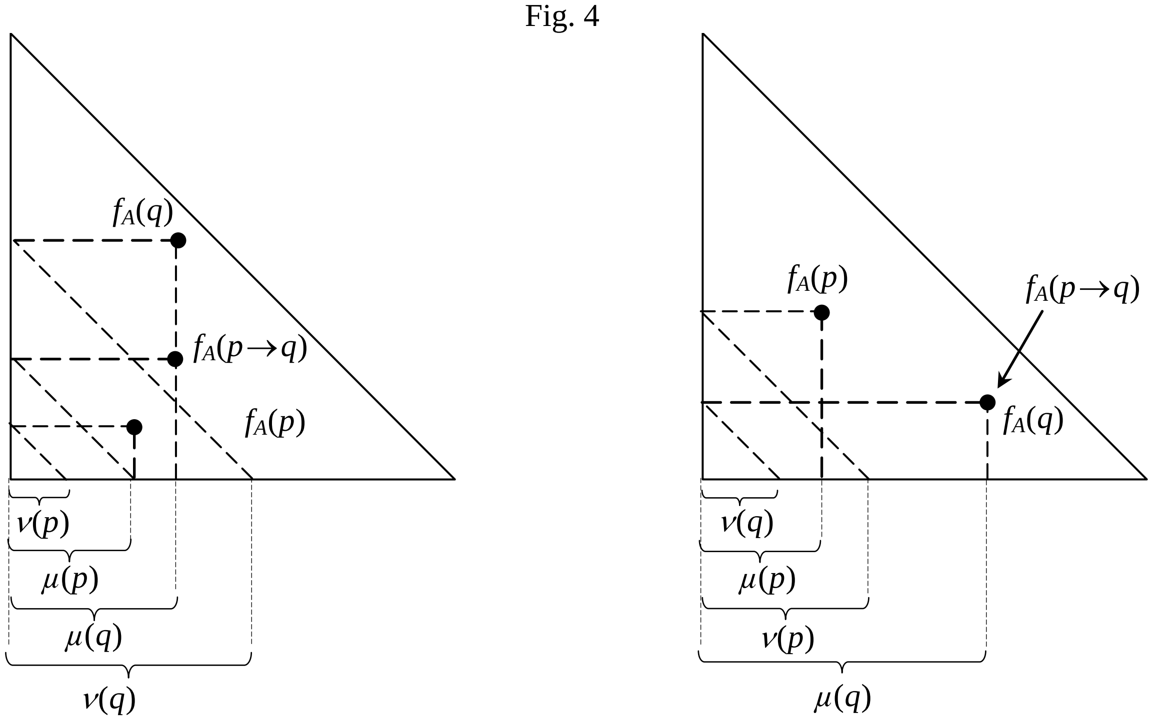
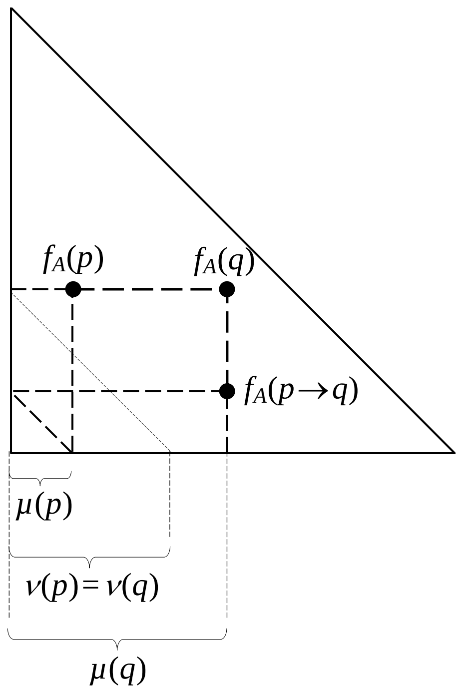
fA(q)
fA(p)
fA(p→q)

(g) (h)

Fig. 4

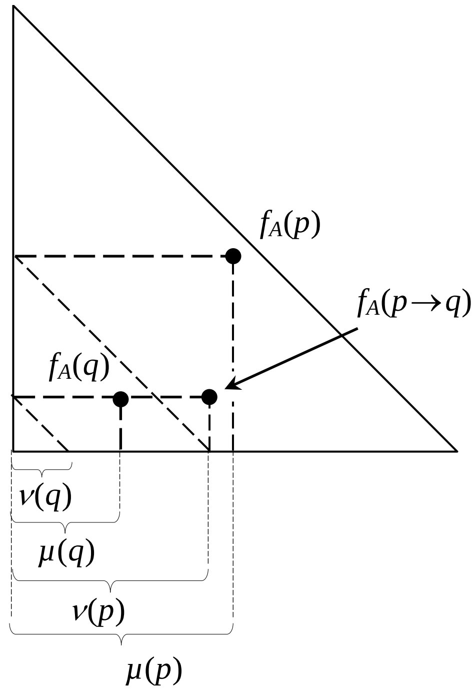

fA(q)
fA(p→q)
fA(p)
ν(p)
μ(p)
μ(q)
ν(q)
fA(p)
fA(p→q)
fA(q)
ν(q)
μ(p)
ν(p)
μ(q)

(a) (b)



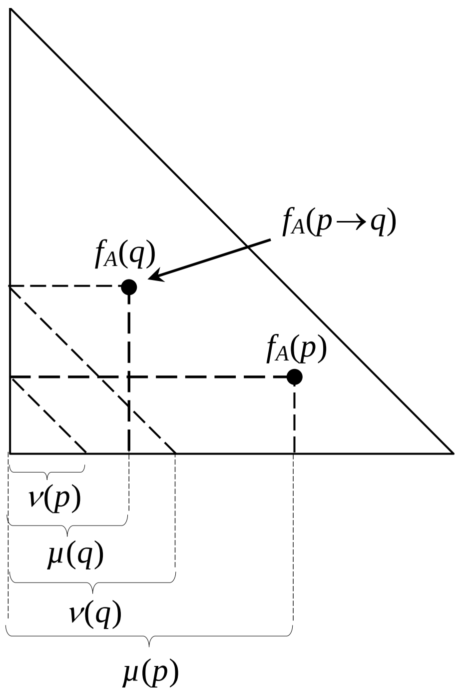
$f_A(p)$
$f_A(q)$
$f_A(p \to q)$
$\mu(p)$
$\nu(p) = \nu(q)$
$\mu(q)$

(c)



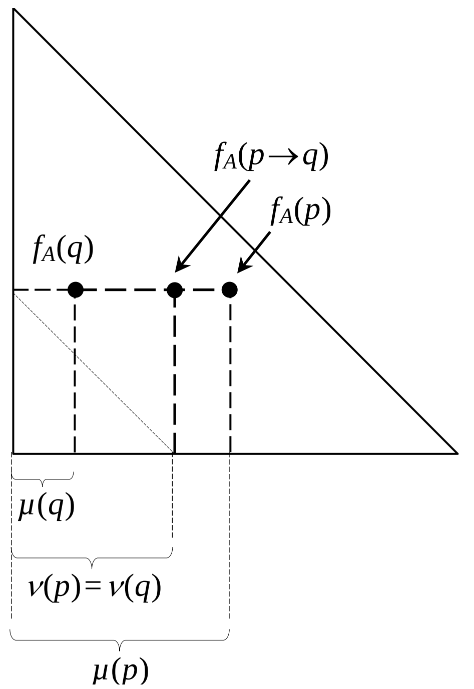
$f_A(p)$
$f_A(p \to q)$
$f_A(q)$
$\nu(q)$
$\mu(q)$
$\nu(p)$
$\mu(p)$

(d)



$f_A(p \to q)$
$f_A(q)$
$f_A(p)$
$\nu(p)$
$\mu(q)$
$\nu(q)$
$\mu(p)$

(e)



$f_A(p \to q)$
$f_A(p)$
$f_A(q)$
$\mu(q)$
$\nu(p) = \nu(q)$
$\mu(p)$

(f)

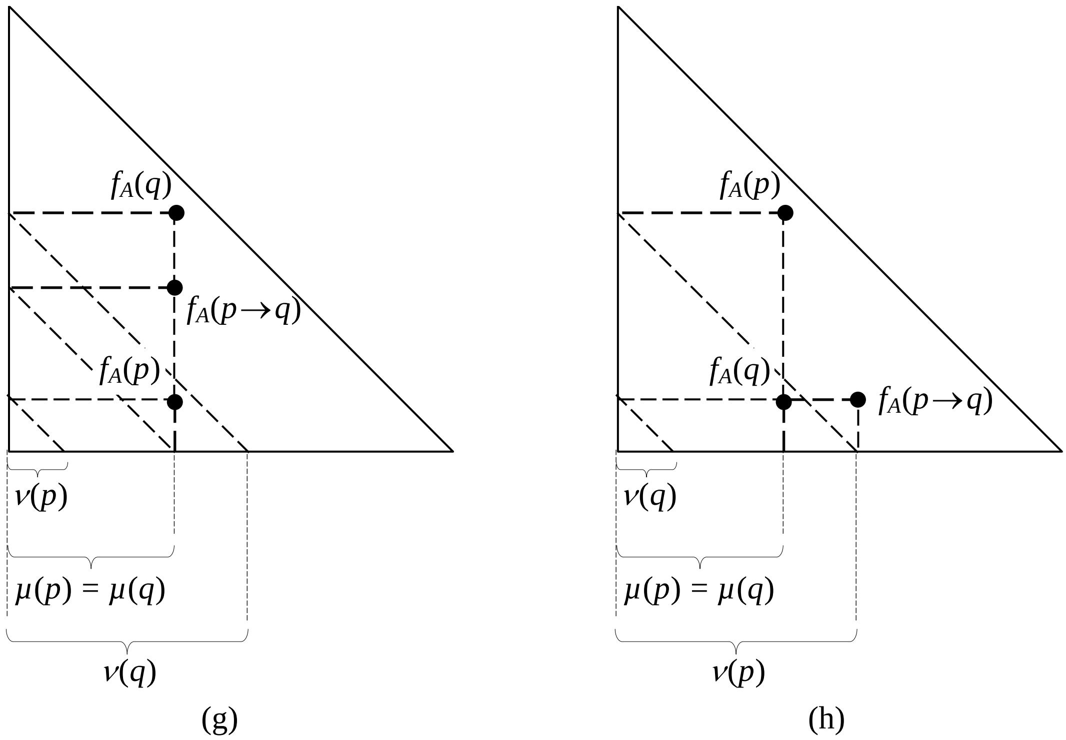


Fig. 5

Below we shall give one by one the geometrical interpretations of the different operators as follows:

- the operator □ has the form from Fig. 6;
- the operator ◊ has the form from Fig. 7;

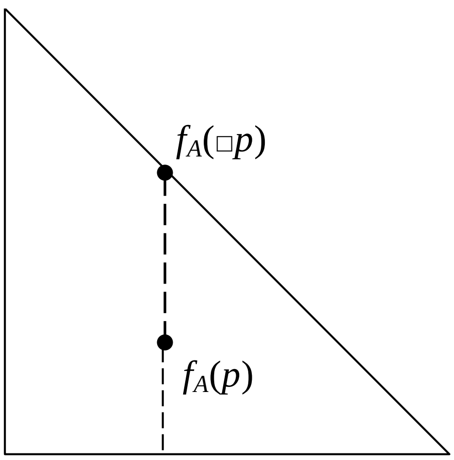


Fig. 6

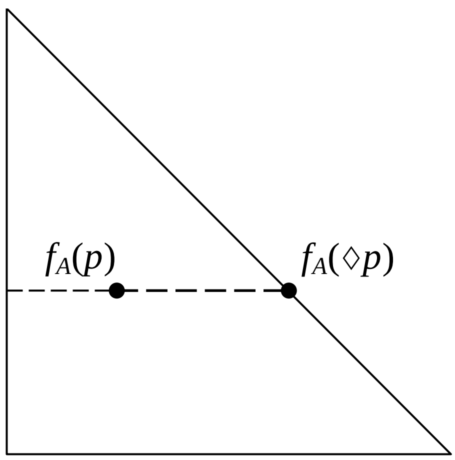


Fig. 7

- the operator D_α assigns to the proposition p a point on the segment between the points p, $\Box p$ and $\Diamond p$ depending on the value of the argument $\alpha \in [0, 1]$ (see Fig. 8);
- the operator $F_{\alpha,\beta}$ assigns to the proposition p a point on the triangle with vertices p, $\Box p$ and $\Diamond p$ depending on the value of the arguments $\alpha, \beta \in [0, 1]$ for which $\alpha + \beta \leq 1$ (see Fig. 9);

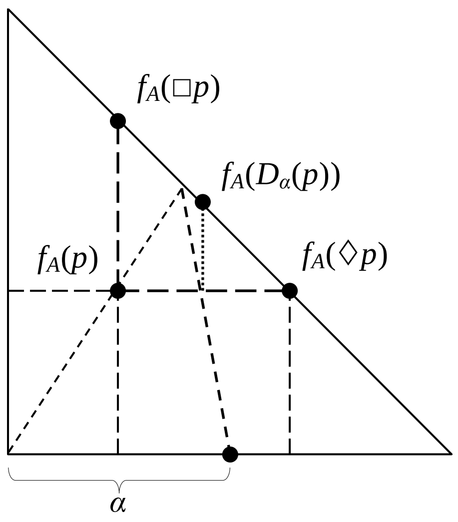


Fig. 8

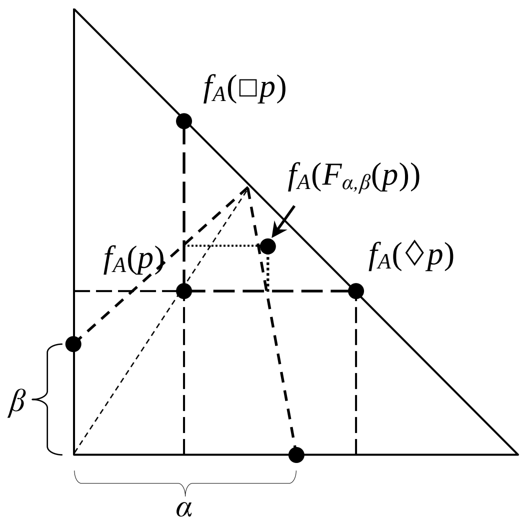


Fig. 9

- the operator $G_{\alpha,\beta}$ assigns to the proposition p a point on the rectangle with vertices p, pr_1p, pr_2p and O, where pr_ip is the i-th projection (i = 1, 2) of the point p, depending on the value of the arguments α, $\beta \in [0, 1]$ (see Fig. 10);

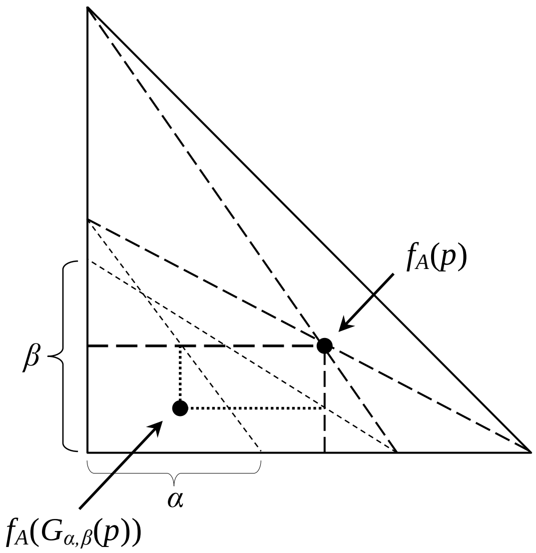


Fig. 10

- the operator $H_{\alpha,\beta}$ assigns to the proposition p a point on the rectangle with vertices pr_2p, p, $\Box p$, and $pr_2\Box p$, depending on the value of the arguments α, $\beta \in [0, 1]$ (see Fig. 11);
- the operator $\bar{H}_{\alpha,\beta}$ assigns to the proposition p a point on the figure with vertices pr_2p, p, $\Box p$, and point $\langle 0, 1\rangle$, depending on the value of the arguments α, $\beta \in [0, 1]$ (see Fig. 12);

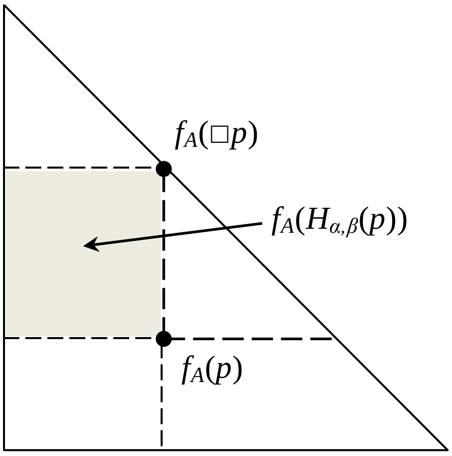


Fig. 11

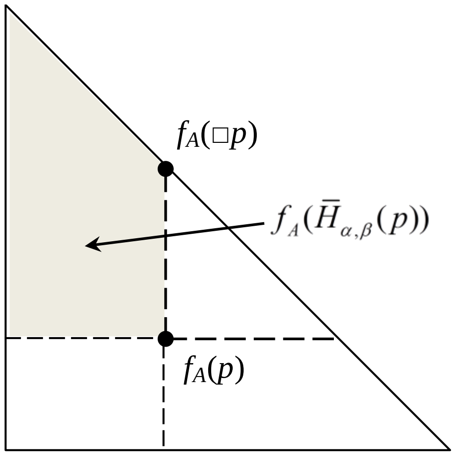


Fig. 12

- the operator $J_{\alpha,\beta}$ assigns to the proposition p a point on the rectangle with vertices pr_1p, p, $\Diamond p$, and $pr_1\Diamond p$, depending on the value of the arguments α, $\beta \in [0, 1]$ (see Fig. 13);
- the operator $\overline{J}_{\alpha,\beta}$ assigns to the proposition p a point on the rectangle with vertices pr_1p, p, $\Diamond p$, and point $\langle 1, 0\rangle$ depending on the value of the arguments α, $\beta \in [0, 1]$ (see Fig. 14).

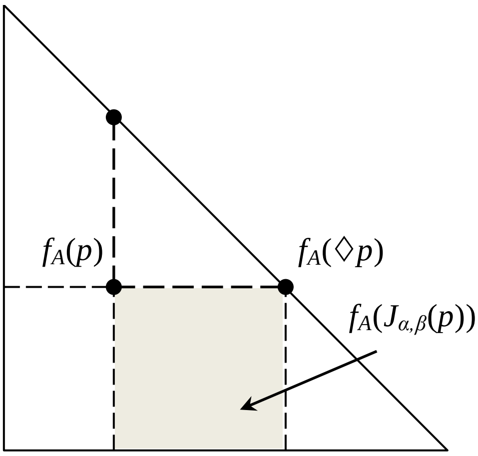


Fig. 13

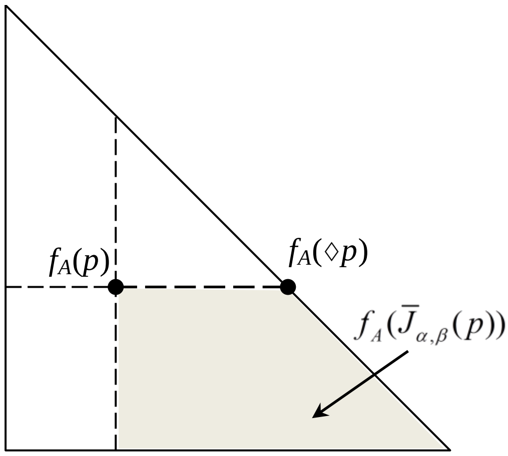


Fig. 14

The above interpretations are the same for the elements of IFPC and IFML as for the elements of IFS. The following two interpretations are related only to the elements of IFSs.

If A, B are two IFSs over E, then function f_{A+B} will assign a point $f_{A+B}(x) \in F$ with coordinates $\langle \mu_A(x) + \mu_B(x) - \mu_A(x).\mu_B(x),\ \nu_A(x).\nu_B(x)\rangle$. There exists only one geometrical interpretation of this operation (see Fig. 15a).

On the Fig. 15b and 15c are shown the constructions of $\mu_A(x).\mu_A(y)$, $\nu_A(x).\nu_A(y)$, respectively.

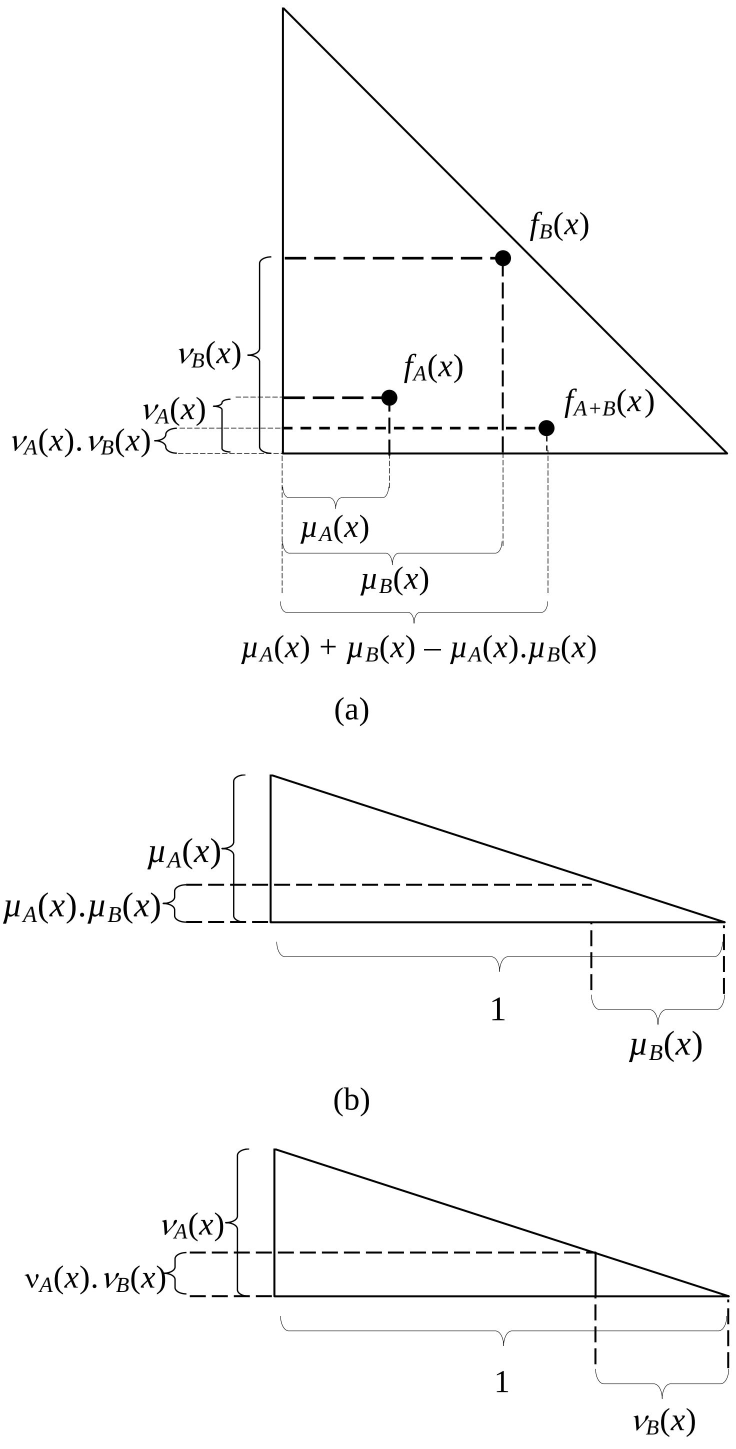


Fig. 15

If A, B are two IFSs over E, then function $f_{A.B}$ will assign to $x \in E$ a point $f_{A.B}(x) \in F$ with coordinates $\langle \mu_A(x).\mu_B(x),\ \nu_A(x) + \nu_B(x) - \nu_A(x).\nu_B(x) \rangle$. Here also there exists only one geometrical interpretation of the operation (see Fig. 16; the constructions of $\mu_A(x).\mu_A(y)$ and $\nu_A(x).\nu_A(y)$ are as above).

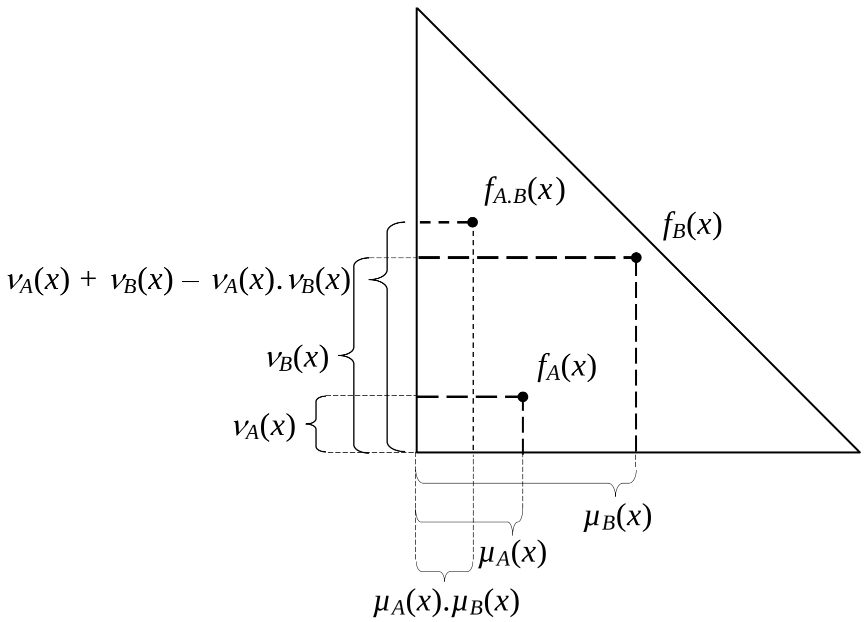


Fig. 16

The inequality $0 \leq a + b \leq 1$ between the coordinates of the point $f_A(p) \in F$ can change to the inequation $0 \leq a^2 + b^2 \leq 1$.

This fact corresponds to the following modification of the IFS, which can be constructed: for every element $x \in E$, the values of the degree of membership and the degree of non-membership satisfy the inequation $0 \leq \mu_A(x)^2 + \nu_A(x)^2 \leq 1$ about some set $A \subset E$.

Obviously, for all real numbers $a, b \in [0; 1]$, if

$0 \leq a + b \leq 1$,

then

$0 \leq a^2 + b^2 \leq 1$.

Hence for the new generated IFS, the above defined operations, relations and operators are valid.

Below we shall show the geometrical interpretation of the operators over the second type IFS, because the geometrical interpretation of the operations over them is almost identical with the above ones and the different is only to the form of the figure F. The new form is shown in Fig. 17. The new interpretation function is noted with g_A, and $g_A : E \to F$.

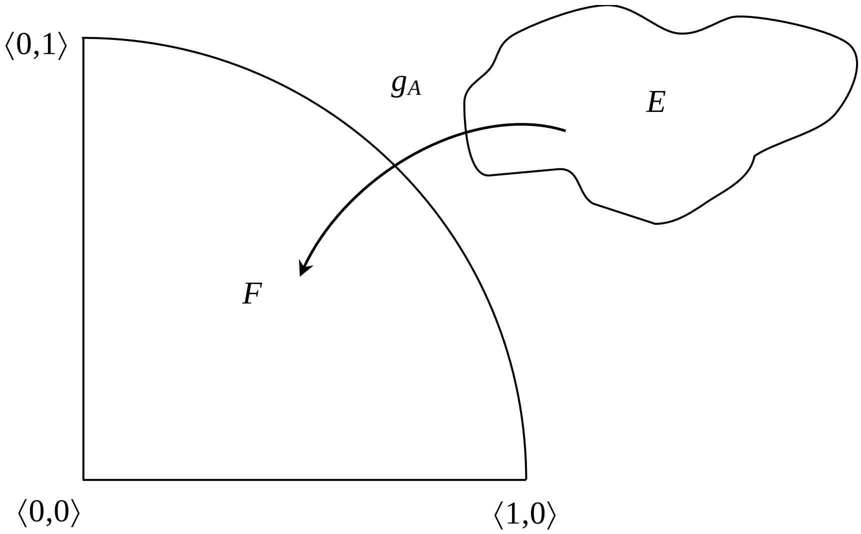


Fig. 17

Now the operators are interpreted as:

- the operator □:

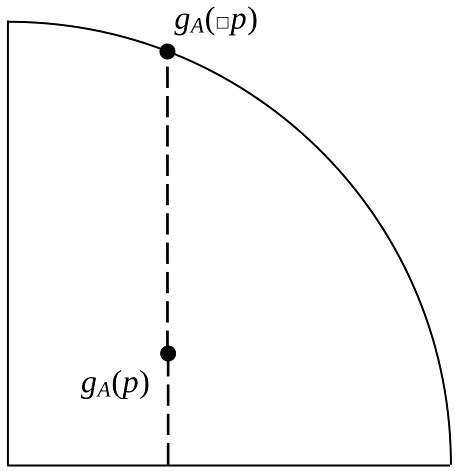


Fig. 18

- the operator ◊:

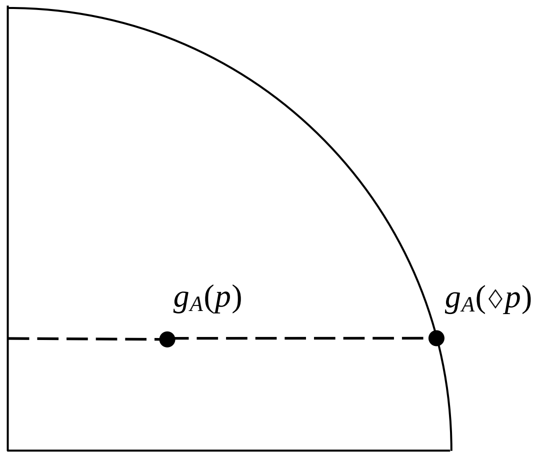


Fig. 19

- the operator D_α:

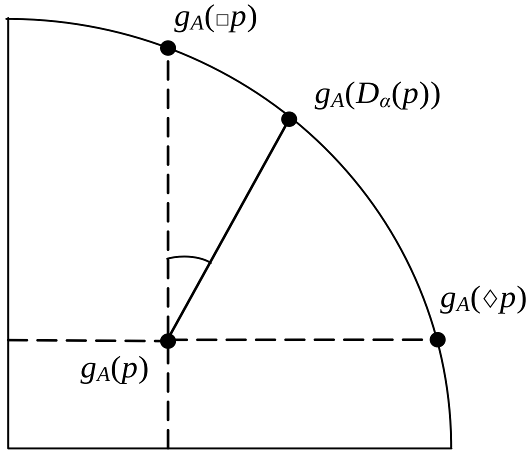


Fig. 20

- the operator $F_{\alpha,\beta}$:

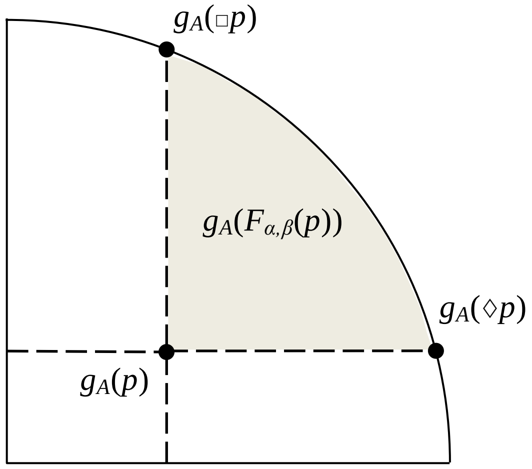


Fig. 21

The operators $G_{\alpha,\beta}$, $H_{\alpha,\beta}$ and $J_{\alpha,\beta}$ have interpretations as above with exactness to the form of the figure F. The remaining operators are interpreted as:

- the operator $\bar{H}_{\alpha,\beta}$:

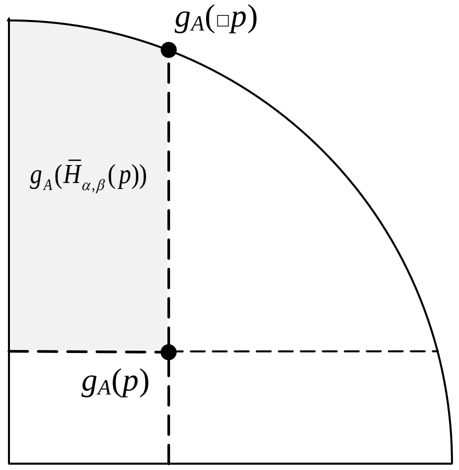


Fig. 22

- the operator $\bar{J}_{\alpha,\beta}$:

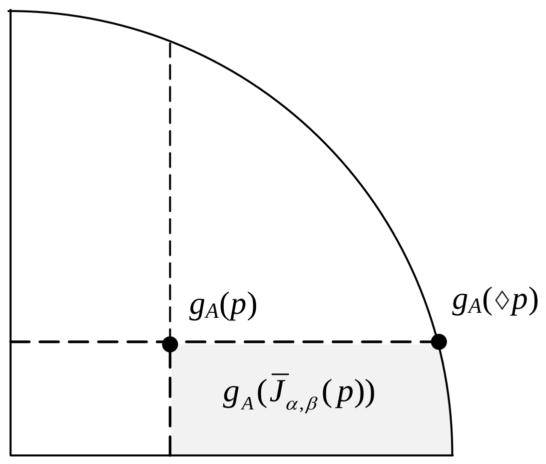


Fig. 23

These interpretations give an illustrative idea of the elements of IFS, IFPC and FMC and the operations and operators over them.

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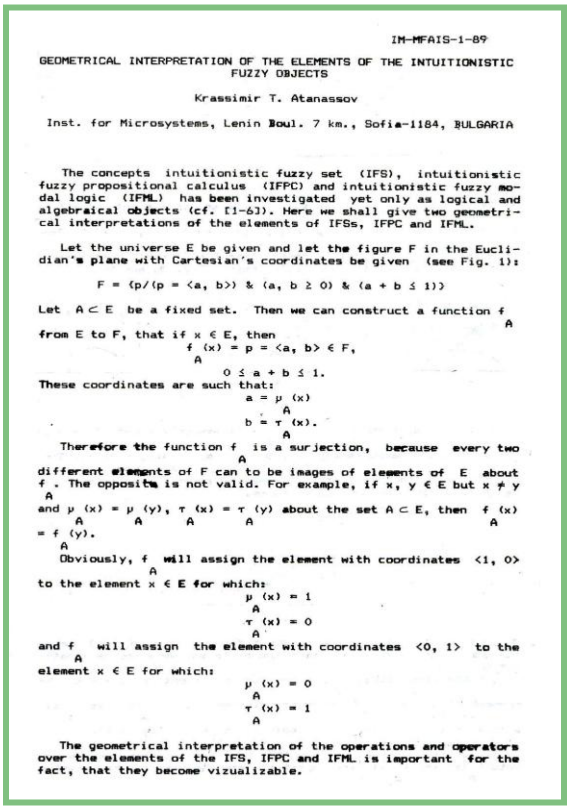

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GEOMETRICAL INTERPRETATION OF THE ELEMENTS OF THE INTUITIONISTIC FUZZY OBJECTS

Krassimir T. Atanassov

Inst. for Microsystems, Lenin Boul. 7 km., Sofia-1184, BULGARIA

The concepts intuitionistic fuzzy set (IFS), intuitionistic fuzzy propositional calculus (IFPC) and intuitionistic fuzzy modal logic (IFML) has been investigated yet only as logical and algebraical objects (cf. [1-6]). Here we shall give two geometrical interpretations of the elements of IFSs, IFPC and IFML.

Let the universe E be given and let the figure F in the Euclidian's plane with Cartesian's coordinates be given (see Fig. 1):

$$F = \{p / (p = \langle a, b \rangle) \,\&\, (a, b \geq 0) \,\&\, (a + b \leq 1)\}$$

Let $A \subset E$ be a fixed set. Then we can construct a function f_A from E to F, that if $x \in E$, then

$$f_A(x) = p = \langle a, b \rangle \in F,$$

$$0 \leq a + b \leq 1.$$

These coordinates are such that:

$$a = \mu_A(x)$$

$$b = \tau_A(x).$$

Therefore the function f_A is a surjection, because every two different elements of F can to be images of elements of E about f_A. The opposite is not valid. For example, if $x, y \in E$ but $x \neq y$ and $\mu_A(x) = \mu_A(y)$, $\tau_A(x) = \tau_A(y)$ about the set $A \subset E$, then $f_A(x) = f_A(y)$.

Obviously, f_A will assign the element with coordinates $\langle 1, 0 \rangle$ to the element $x \in E$ for which:

$$\mu_A(x) = 1$$

$$\tau_A(x) = 0$$

and f_A will assign the element with coordinates $\langle 0, 1 \rangle$ to the element $x \in E$ for which:

$$\mu_A(x) = 0$$

$$\tau_A(x) = 1$$

The geometrical interpretation of the operations and operators over the elements of the IFS, IFPC and IFML is important for the fact, that they become vizualizable.

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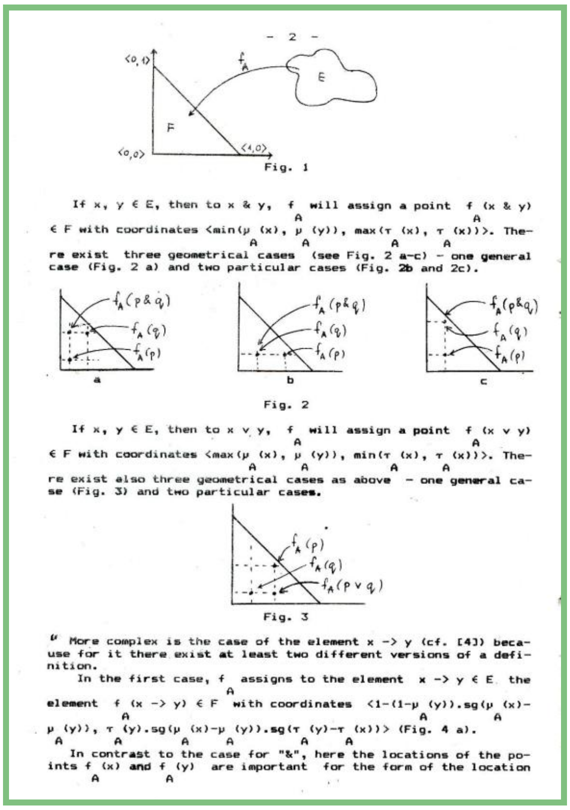


- 2 -

Fig. 1

If $x, y \in E$, then to $x \,\&\, y$, f_A will assign a point $f_A(x \,\&\, y) \in F$ with coordinates $\langle \min(\mu_A(x), \mu_A(y)), \max(\tau_A(x), \tau_A(x)) \rangle$. There exist three geometrical cases (see Fig. 2 a-c) - one general case (Fig. 2 a) and two particular cases (Fig. 2b and 2c).

Fig. 2

If $x, y \in E$, then to $x \vee y$, f_A will assign a point $f_A(x \vee y) \in F$ with coordinates $\langle \max(\mu_A(x), \mu_A(y)), \min(\tau_A(x), \tau_A(x)) \rangle$. There exist also three geometrical cases as above - one general case (Fig. 3) and two particular cases.

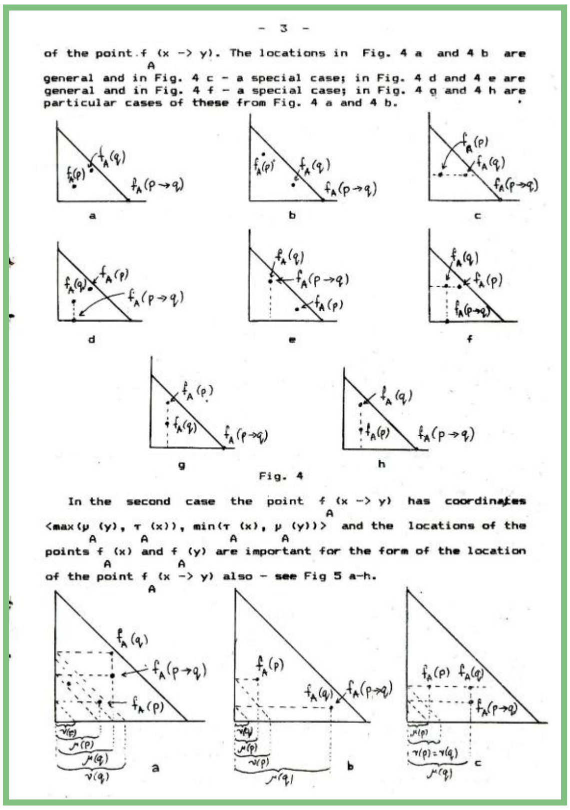
Fig. 3

More complex is the case of the element $x \rightarrow y$ (cf. [4]) because for it there exist at least two different versions of a definition.

In the first case, f_A assigns to the element $x \rightarrow y \in E$ the element $f_A(x \rightarrow y) \in F$ with coordinates $\langle 1-(1-\mu_A(y)).sg(\mu_A(x)-\mu_A(y)), \tau_A(y).sg(\mu_A(x)-\mu_A(y)).sg(\tau_A(y)-\tau_A(x)) \rangle$ (Fig. 4 a).

In contrast to the case for "&", here the locations of the points $f_A(x)$ and $f_A(y)$ are important for the form of the location

Page 2



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of the point $f_A(x \rightarrow y)$. The locations in Fig. 4 a and 4 b are general and in Fig. 4 c - a special case; in Fig. 4 d and 4 e are general and in Fig. 4 f - a special case; in Fig. 4 g and 4 h are particular cases of these from Fig. 4 a and 4 b.

Fig. 4

In the second case the point $f_A(x \rightarrow y)$ has coordinates $\langle \max(\mu_A(y), \tau_A(x)), \min(\tau_A(x), \mu_A(y)) \rangle$ and the locations of the points $f_A(x)$ and $f_A(y)$ are important for the form of the location of the point $f_A(x \rightarrow y)$ also - see Fig 5 a-h.

Page 3

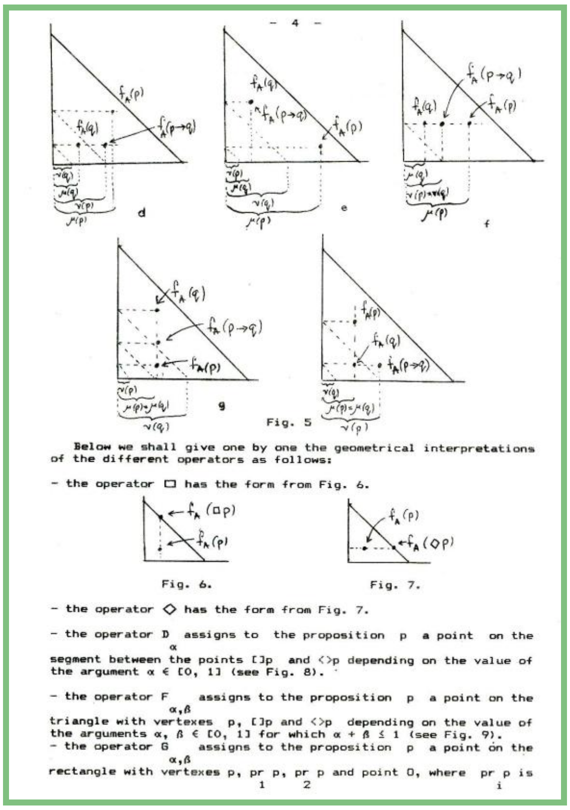

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Fig. 5

Below we shall give one by one the geometrical interpretations of the different operators as follows:

- the operator □ has the form from Fig. 6.

Fig. 6. Fig. 7.

- the operator ◇ has the form from Fig. 7.
- the operator D_α assigns to the proposition p a point on the segment between the points []p and <>p depending on the value of the argument $\alpha \in [0, 1]$ (see Fig. 8).
- the operator $F_{\alpha,\beta}$ assigns to the proposition p a point on the triangle with vertexes p, []p and <>p depending on the value of the arguments $\alpha, \beta \in [0, 1]$ for which $\alpha + \beta \leq 1$ (see Fig. 9).
- the operator $G_{\alpha,\beta}$ assigns to the proposition p a point on the rectangle with vertexes p, pr_1 p, pr_2 p and point O, where pr_i p is

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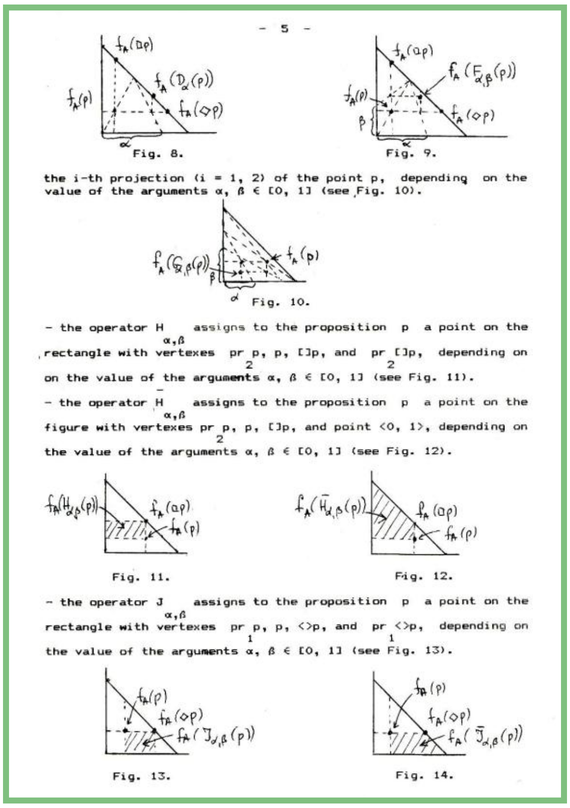

– 5 –

Fig. 8.

Fig. 9.

the i-th projection (i = 1, 2) of the point p, depending on the value of the arguments $\alpha, \beta \in [0, 1]$ (see Fig. 10).

Fig. 10.

– the operator $H_{\alpha,\beta}$ assigns to the proposition p a point on the rectangle with vertexes $pr_2 p$, p, []p, and pr_2 []p, depending on on the value of the arguments $\alpha, \beta \in [0, 1]$ (see Fig. 11).

– the operator $\bar{H}_{\alpha,\beta}$ assigns to the proposition p a point on the figure with vertexes $pr_2 p$, p, []p, and point <0, 1>, depending on the value of the arguments $\alpha, \beta \in [0, 1]$ (see Fig. 12).

Fig. 11.

Fig. 12.

– the operator $J_{\alpha,\beta}$ assigns to the proposition p a point on the rectangle with vertexes $pr_1 p$, p, <>p, and pr_1 <>p, depending on the value of the arguments $\alpha, \beta \in [0, 1]$ (see Fig. 13).

Fig. 13.

Fig. 14.

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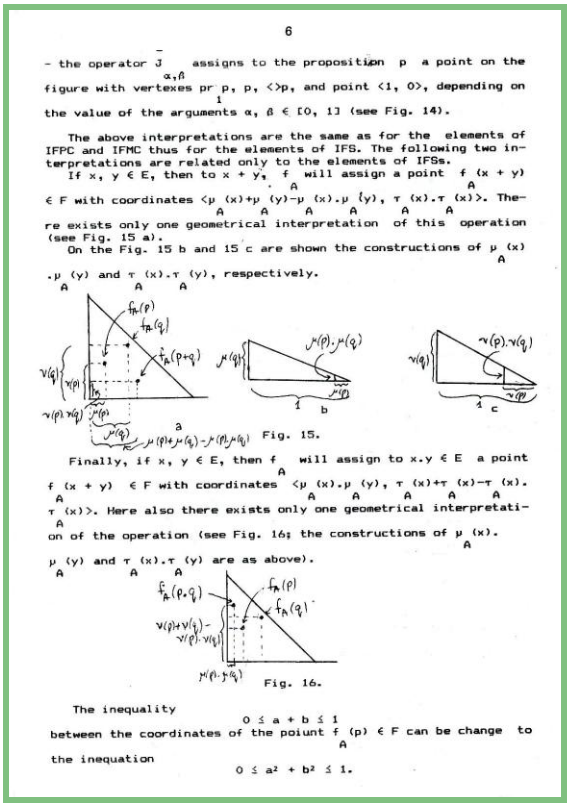

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– the operator $\bar{J}_{\alpha,\beta}$ assigns to the proposition p a point on the figure with vertexes $pr_1 p$, p, <>p, and point <1, 0>, depending on the value of the arguments $\alpha, \beta \in [0, 1]$ (see Fig. 14).

The above interpretations are the same as for the elements of IFPC and IFMC thus for the elements of IFS. The following two interpretations are related only to the elements of IFSs.

If $x, y \in E$, then to x + y, f_A will assign a point $f_A(x + y) \in F$ with coordinates $\langle \mu_A(x)+\mu_A(y)-\mu_A(x).\mu_A(y), \tau_A(x).\tau_A(x) \rangle$. There exists only one geometrical interpretation of this operation (see Fig. 15 a).

On the Fig. 15 b and 15 c are shown the constructions of $\mu_A(x).\mu_A(y)$ and $\tau_A(x).\tau_A(y)$, respectively.

Fig. 15.

Finally, if $x, y \in E$, then f_A will assign to $x.y \in E$ a point $f_A(x + y) \in F$ with coordinates $\langle \mu_A(x).\mu_A(y), \tau_A(x)+\tau_A(x)-\tau_A(x).\tau_A(x) \rangle$. Here also there exists only one geometrical interpretation of the operation (see Fig. 16; the constructions of $\mu_A(x).\mu_A(y)$ and $\tau_A(x).\tau_A(y)$ are as above).

Fig. 16.

The inequality

$$0 \le a + b \le 1$$

between the coordinates of the poiunt $f_A(p) \in F$ can be change to the inequation

$$0 \le a^2 + b^2 \le 1.$$

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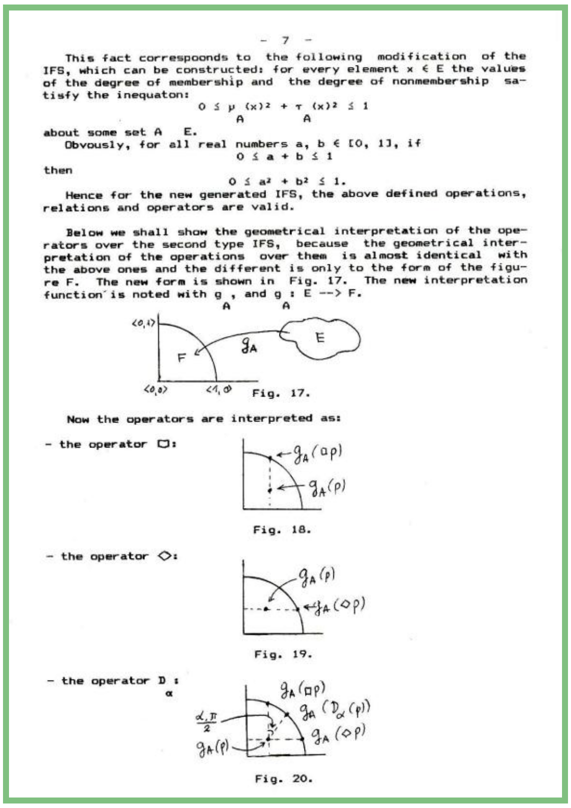

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This fact correspoonds to the following modification of the IFS, which can be constructed: for every element $x \in E$ the values of the degree of membership and the degree of nonmembership satisfy the inequaton:

$$0 \le \mu_A(x)^2 + \tau_A(x)^2 \le 1$$

about some set A E.

Obvously, for all real numbers $a, b \in [0, 1]$, if

$$0 \le a + b \le 1$$

then

$$0 \le a^2 + b^2 \le 1.$$

Hence for the new generated IFS, the above defined operations, relations and operators are valid.

Below we shall show the geometrical interpretation of the operators over the second type IFS, because the geometrical interpretation of the operations over them is almost identical with the above ones and the different is only to the form of the figure F. The new form is shown in Fig. 17. The new interpretation function is noted with g_A, and $g_A : E \to F$.

Fig. 17.

Now the operators are interpreted as:

– the operator □:

Fig. 18.

– the operator ◇:

Fig. 19.

– the operator D_α :

Fig. 20.

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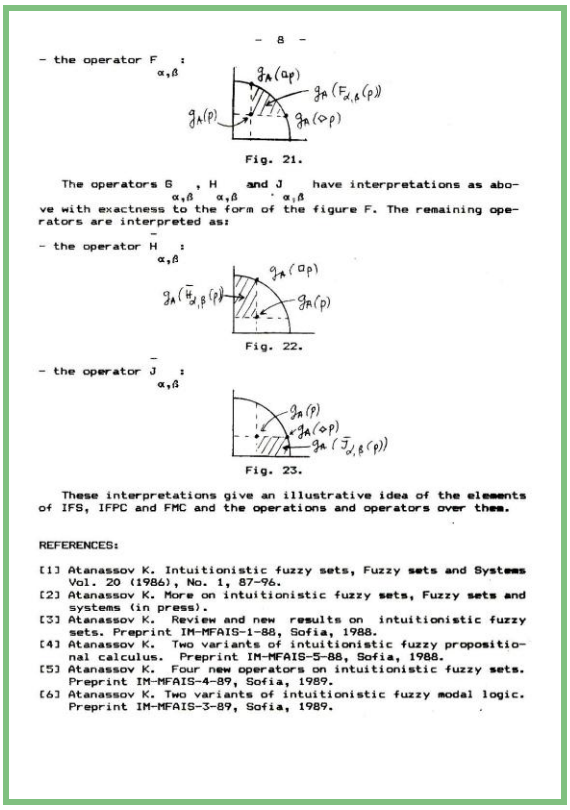

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– the operator $F_{\alpha,\beta}$:

Fig. 21.

The operators $G_{\alpha,\beta}$, $H_{\alpha,\beta}$ and $J_{\alpha,\beta}$ have interpretations as above with exactness to the form of the figure F. The remaining operators are interpreted as:

– the operator $\bar{H}_{\alpha,\beta}$:

Fig. 22.

– the operator $\bar{J}_{\alpha,\beta}$:

Fig. 23.

These interpretations give an illustrative idea of the elements of IFS, IFPC and FMC and the operations and operators over them.

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