Two Variants of Intuitionistic Fuzzy Modal Logics

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How to cite:

Atanassov K. T. Two Variants of Intuitionistic Fuzzy Modal Logics, Mathematical Foundations of Artificial Intelligence Seminar, Sofia, 1989, Preprint IM-MFAIS-3-89. Reprinted: Int J Bioautomation, 2016, 20(S1), S43-S54.


IM-MFAIS-1-89

GEOMETRICAL INTERPRETATION OF THE ELEMENTS OF THE INTUITIONISTIC FUZZY OBJECTS

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The concepts intuitionistic fuzzy set (IFS), intuitionistic fuzzy propositional calculus (IFPC) and intuitionistic fuzzy modal logic (IFML) has been investigated yet only as logical and algebraical objects (cf. [1-4]). Here we shall give two geometrical interpretations of the elements of IFSs, IFPC and IFML.

Following the ideas from [1] and using the notation from there, we shall construct two variants of intuitionistic fuzzy modal logics (IFMLs). The modal logic axioms used are from [2].

1. sg-variant of IFML

For a proposition p for which:

$V(p) = \langle a, b\rangle$

we shall define the following operations (from [1]):

$V(\neg p) = \langle b, a\rangle$

$V(p \,\&\, q) = \langle \min(\mu(p), \mu(q)), \max(\nu(p), \nu(q))\rangle$

$V(p \vee q) = \langle \max(\mu(p), \mu(q)), \min(\nu(p), \nu(q))\rangle$

$V(p \supset q) = \langle 1 - (1 - \mu(q)).\mathrm{sg}(\mu(p) - \mu(q)), \nu(p).(\mu(p) - \mu(q)).\mathrm{sg}(\nu(q) - \nu(p))\rangle$

where

sg = 1, if $x > 0$ and sg = 0, if $x \leq 0$,

and operators (new definitions)

$V(\Box p) = \langle a, 1 - a\rangle$,

$V(\Diamond p) = \langle 1 - b, b\rangle$.

Let the truth value function V be defined such a way that for propositions p, q, r:

$\neg V(p) = V(\neg p)$,

$V(p) \wedge V(q) = V(p \,\&\, q)$,

$V(p) \vee V(q) = V(p \vee q)$,

$V(p) \rightarrow V(q) = V(p \supset q)$,

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$\Box V(p) = V(\Box p)$
$\Diamond V(p) = V(\Diamond p)$

and we shall construct a sg-variant of an IFMC.

Let everywhere:

$V(p) = \langle a, b\rangle$
$V(q) = \langle c, d\rangle$
$V(r) = \langle e, f\rangle$

Initially, we shall prove the following:

Theorem 1.1: The following assertions are tautologies ([2, Paragraph 13.1]).

(a) $(p \vee q) \supset p$,
(b) $p \supset (p \vee q)$,
(c) $(p \vee q) \supset (q \vee p)$,
(d) $(p \supset q) \supset ((r \vee p) \supset (r \vee q))$.

Proof:

(d) $V((p \supset q) \supset ((r \vee p) \supset (r \vee q)))$
$= (\langle a, b\rangle \to \langle c, d\rangle) \to (\langle \max(a, e), \min(b, f)\rangle \to \langle \max(c, e), \min(d, f)\rangle)$
$= \langle 1 - (1 - c).\mathrm{sg}(a - c), d.\mathrm{sg}(a - c), \mathrm{sg}(d - b)\rangle \to \langle 1 - (1 - \max(c, e).\mathrm{sg}(\max(a, e)$
$- \max(c, e)), \min(d, f).\mathrm{sg}(\max(a, e) - \max(c, e)).\mathrm{sg}(\min(d, f) - \min(b, f))\rangle$
$= \langle 1 - (1 - \max(c, e)).\mathrm{sg}(\max(a, e) - \max(c, e)).\mathrm{sg}((1 - \max(c, e)).\mathrm{sg}(\max(a, e)$
$- \max(c, e)) - (1 - c).\mathrm{sg}(a - c), \min(d, f).\mathrm{sg}(\max(a, e) - \max(c, e)).\mathrm{sg}(\min(d, f)$
$- \min(b, f)).\mathrm{sg}((1 - \max(c, e)).\mathrm{sg}(\max(a, e) - \max(c, e)) - (1 - c).\mathrm{sg}(a - c)).\mathrm{sg}(\min(d,$
$f).\mathrm{sg}(\max(a, e) - \max(c, e)).\mathrm{sg}(\min(d, f) - \min(b, f)) - d.\mathrm{sg}(a - c).\mathrm{sg}(d - b))\rangle$

if $a \leq c$, from $\mathrm{sg}(\max(a, e) - \max(c, e)) = 0$:

$= \langle 1, 0\rangle$

if $a > c$, then:

$= \langle 1 - (1 - \max(c, e)).\mathrm{sg}(\max(a, e) - \max(c, e)).\mathrm{sg}((1 - \max(c, e)).\mathrm{sg}(\max(a, e)$
$- \max(c, e)) - 1 + c), \min(d, f).\mathrm{sg}(\max(a, e) - \max(c, e)).\mathrm{sg}(\min(d, f)$
$- \min(b, f)).\mathrm{sg}((1 - \max(c, e)).\mathrm{sg}(\max(a, e) - \max(c, e)) - 1 + c).\mathrm{sg}(\min(d,$
$f).\mathrm{sg}(\max(a, e) - \max(c, e)).\mathrm{sg}(\min(d, f) - \min(b, f)) - d.\mathrm{sg}(d - b))\rangle$

if $e \geq a$, (hence $e > c$), from $\mathrm{sg}(\max(a, e) - \max(c, e)) = 0$:

$= \langle 1, 0\rangle$
if $a > e$, from $\max(c,e) < a$ and $\mathrm{sg}(c - \max(c,e)) = 0$:

$= \langle 1-(1-\max(c,e)).\mathrm{sg}(c-\max(c,e)), \min(d,f).\mathrm{sg}(\min(d,f), \min(b,f)).$
$\mathrm{sg}(c-\max(c,e)).\mathrm{sg}(\min(d,f).\mathrm{sg}(\max(a,e)-\max(c,e)).\mathrm{sg}(\min(d,f)-\min(b,f))$
$-d.\mathrm{sg}(d-b))\rangle$
$= \langle 1, 0\rangle.$

(a)-(c) are proved analogically. □

Theorem 1.2:

(a) If p is a tautology, then $\Box p$ is also tautology.

(b) $\Box(p \supset q) \supset (\Box p \supset \Box q)$ is a tautology.

(c) $\Box p \supset p$ is a tautology.

Proof: (a) From the condition that p is a tautology follows that $V(p) = \langle 1, 0\rangle$. Hence, $V(\Box p) = \langle 1, 0\rangle$, i.e., $\Box p$ is a tautology.

(b) $\Box(p \supset q) \supset (\Box p \supset \Box q)$
$= \Box\langle 1-(1-c).\mathrm{sg}(a-c), d.\mathrm{sg}(a-c).\mathrm{sg}d-b\rangle \to (\langle a, 1-a\rangle \to \langle c, 1-c\rangle$
$= \langle 1-(1-c).\mathrm{sg}(a-c), (1-c).\mathrm{sg}(a-c)\rangle \to \langle 1-(1-c).\mathrm{sg}(a-c), (1-c).\mathrm{sg}(a-c)^2\rangle$
$= \langle 1-(1-c).\mathrm{sg}(a-c).\mathrm{sg}(0), (1-c).\mathrm{sg}(a-c).\mathrm{sg}(0)^2\rangle$
$= \langle 1, 0\rangle.$

(c) is proved analogically. □

Let for a given propositional form A (c.f. [1, 3]):

$\Diamond A$ denote $\neg\Box\neg A$,

$A \Rightarrow B$ denote $\Box(A \supset B)$,

$A \Leftrightarrow B$ denote $\Box(A \equiv B)$,

where $A \equiv B$ denotes $(A \supset B)\&(B \supset A)$.

Theorem 1.3: The following assertions are tautologies ([2, Paragraphs 24.0 and 24.1]):

(a) $\neg\Box P \equiv \Diamond\neg P$,

(b) $\Box P \equiv \neg\Diamond\neg P$,

(c) $\neg\Diamond P \equiv \Box\neg P$,

(d) $\Diamond P \equiv \neg\Box\neg P$,

(e) $P \supset \Diamond P$,

(f) $\Box P \supset \Diamond P$.

Proof: (a) $V(\Diamond\Box P \equiv \Diamond\Diamond P)$
$= (\langle 1-a, a\rangle \to \langle 1-a, a\rangle)\&(\langle 1-a, a\rangle \to \langle 1-a, a\rangle)$
$= \langle 1-a.\text{sg}(0), a.\text{sg}(0)^2\rangle$
$= \langle 1, 0\rangle$.

The other assertions are proved analogically. □

Theorem 1.4: The following assertions are tautologies ([2, Paragraphs 24.2 and 24.3]):

(a) $\Box(p\&q) \equiv (\Box p\&\Box q)$,

(b) $(\Box p \vee \Box q) \supset \Box(p \vee q)$,

(c) $\Diamond(p \vee q) \equiv (\Diamond p \vee \Diamond q)$,

(d) $\Diamond(p\&q) \supset (\Diamond p\&\Diamond q)$,

(e) $\Diamond(p\&q) \supset \Diamond p$,

(f) $(p \Rightarrow q) \supset (\Box p \supset \Box q)$,

(g) $((p \Rightarrow q)\&\Box p) \supset \Box q$,

(h) $(p \Leftrightarrow q) \supset (\Box p \equiv \Box q)$.

Proof: (a) $V(\Box(p\&q) \equiv (\Box p\&\Box q))$
$= (\Box\langle \min(a,c), \max(b,d)\rangle \to \langle \min(a,c), \max(1-a, 1-c)\rangle)\&$
$(\langle \min(a,c), \max(1-a, 1-c)\rangle \to \Box\langle \min(a,c), \max(b,d)\rangle)$
$= (\Box\langle \min(a,c), 1-\min(a,c)\rangle \to \langle \min(a,c), 1-\min(a,c)\rangle)\&$
$(\langle \min(a,c), 1-\max(a,c)\rangle \to \Box\langle \min(a,c), 1-\min(a,c)\rangle)$
$= \langle 1, 0\rangle$.

(b)-(h) are proved analogically. □

Theorem 1.5: The following assertions are valid ([2, Paragraph 24.4]):

(a) If $A \Rightarrow B$ is a tautology, then $\Box A \supset \Box B$ is a tautology.

(b) If $A \Leftrightarrow B$ is a tautology, then $\Box A \equiv \Box B$ is a tautology.

Proof: (a) Let for a given A and B : $V(A) = \langle a, b\rangle$, $V(B) = \langle c, d\rangle$, $A \Rightarrow B$ is a tautology, i.e., $V(A \Rightarrow B) = \langle 1, 0\rangle$. Hence:

$\Box(\langle a, b\rangle \to \langle c, d\rangle) = \langle 1, 0\rangle$,

i.e., $1-(1-c).\text{sg}(a-c) = 1$, i.e., $c = 1$ or $a \leq c$. Therefore $a \leq c$. Then:

$V(\Box A \supset \Box B) = \langle a, 1-a\rangle \to \langle c, 1-c\rangle$
$= \langle 1-(1-c).\text{sg}(a-c), (1-c).\text{sg}(a-c)^2$
$= \langle 1, 0\rangle$.

(b) is proved analogically. □

2. (max-min)-variant of IFML

Here we shall save all notations from [1], without the notation of the implication. For the last notation here we shall use the definition from [1, Paragraph 2]:

$$V(p \supset q) = \langle \max(\nu(p), \mu(q)), \min(\mu(p), \nu(q)) \rangle,$$

and also:

$$V(p) \to V(q) = V(p \supset q).$$

Here we shall use the definition of intuitionistic fuzzy tautology (IFT) from [1]:

p is an IFT iff $V(p) = \langle a, b \rangle$ and $a \geq b$.

The theorems from [1] will be again proved for the new variant.

Theorem 2.1: The following assertions are IFTs:

(a) $(p \vee q) \supset p$,

(b) $p \supset (p \vee q)$,

(c) $(p \vee q) \supset (q \vee p)$,

(d) $(p \supset q) \supset ((r \supset p) \supset (r \supset q)))$.

Proof: (d) $V((p \supset q) \supset ((r \supset p) \supset (r \supset q))))$
$= (\langle a, b \rangle \to \langle c, d \rangle) \to (\langle \max(a, e), \min(b, f) \rangle \to \langle \max(c, e), \min(d, f) \rangle)$
$= \langle \max(b, d), \min(a, d) \rangle \to \langle \max(c, e, \min(b, f)), \min(d, f, \max(a, e)) \rangle$
$= \langle \max(\min(a, d), c, e, \min(b, f)), \min(d, f, \max(a, e), \max(b, d)) \rangle$

and

$\max(\min(a, d), c, e, \min(b, f)) - \min(d, f, \max(a, e), \max(b, d))$
$\geq \max(\min(a, d), c, e) - \min(d, \max(a, e))$

if $a \geq d$

$= \max(d, c, e) - \min(d, \max(a, e)) \geq 0$

if $a < d$

$= \max(a, c, e) - \min(d, \max(a, e))$
$\geq \max(a, e) - \min(d, \max(a, e)) \geq 0.$

(a)-(c) are proved analogically. □

Theorem 2.2:

(a) $\Box(p \supset q) \supset (\Box p \supset \Box q)$ is an IFT,

(b) $\Box p \supset p$ is an IFT.

Proof: (a) $V(\Box(p \supset q) \supset (\Box p \supset \Box q))$
$= \Box\langle \max(b,c), \min(a,d)\rangle \supset (\langle a, 1-a\rangle \supset \langle c, 1-c\rangle)$
$= \langle \max(b,c), 1-\max(b,c)\rangle \supset \langle \max(1-a,c), \min(a,1-c)\rangle$
$= \langle \max(1-\max(b,c), 1-a, c), \min(a, 1-c, \max(b,c))\rangle$

and

$\max(1-\max(b,c), 1-a, c) - \min(a, 1-c.\max(b,c))$
$= 2(1-\max(b,c), 1-a, c) - 1.$

Let us assume that:

$\max(1-\max(b,c), 1-a, c) < \frac{1}{2}.$

Hence: $\max(b,c) > \frac{1}{2}, 1-a < \frac{1}{2}$ and $c < \frac{1}{2}$. Therefore, $b > \frac{1}{2}$ and $a > \frac{1}{2}$ which is a contradiction. Hence,

$\max(1-\max(b,c), 1-a, c) \geq \frac{1}{2},$

with which (a) is proved.

(b) is proved directly. □

Theorem 2.3: The following assertions are IFTs:

(a) $\neg\Box p \equiv \Diamond\neg p$,

(b) $\Box p \equiv \neg\Diamond\neg p$,

(c) $\neg\Diamond p \equiv \Box\neg p$,

(d) $\Diamond p \equiv \neg\Box\neg p$,

(e) $p \supset \Diamond p$,

(f) $\Box p \supset \Diamond p$.

Proof: (a) $V(\neg\Box p \equiv \Diamond\neg p)$
$= (\neg\langle a, 1-a\rangle \to \Box\langle b,a\rangle)\&(\Box\langle b,a\rangle \to \neg\langle a, 1-a\rangle)$
$= (\langle 1-a, a\rangle \to \langle 1-a, a\rangle)\&(\langle 1-a, a\rangle \to \langle 1-a, a\rangle)$
$= \langle \max(a, 1-a), \min(a, 1-a)\rangle$, which is an IFS.

(b)-(f) are proved analogically. □

Theorem 2.4: The following assertions are IFTs:

(a) $\Box(p\&q) \equiv (\Box p\&\Box q)$,
(b) $\Box(p \vee q) \supset \Box(p \vee q)$,
(c) $\Diamond(p \vee q) \equiv (\Diamond p \vee \Diamond q)$,
(d) $\Diamond(p\&q) \supset (\Diamond p\&\Diamond q)$,
(e) $\Diamond(p\&q) \supset \Diamond p$,
(f) $(p \Rightarrow q) \supset (\Box p \supset \Box q)$,
(g) $((p \Rightarrow q)\&\Box p) \supset \Box q$,
(h) $(p \Rightarrow q) \supset (\Diamond p \supset \Diamond q)$,
(i) $((p \Rightarrow q)\&\Diamond p) \supset \Diamond q$,
(j) $(p \Leftrightarrow q) \supset (\Box p \equiv \Box q)$,
(k) $(p \Leftrightarrow q) \supset (\Diamond p \equiv \Diamond q)$.

Proof: (a) $V(\Box(p\&q) \equiv (\Box p\&\Box q))$
$= (\Box\langle \min(a,c), \max(b,d)\rangle \to (\langle a, 1-a\rangle \wedge \langle c, 1-c\rangle)) \wedge ((\langle a, 1-a\rangle \wedge$
$\langle c, 1-c\rangle) \to \Box\langle \min(a,c), \max(b,d)\rangle)$
$= (\Box\langle \min(a,c), 1-\min(a,c)\rangle \to \langle \min(a,c), \max(1-a, 1-c)\rangle) \wedge$
$(\langle \min(a,c), \max(1-a, 1-c)\rangle \to \langle \min(a,c), 1-\min(a,c)\rangle)$
$= \langle \max(\min(1-\min(a,c), \min(a,c)), \min(a,c,\max(1-a,1-c))\rangle \wedge$
$\langle \max(\max(1-a,1-c,\min(a,c)), \min(a,c,1-\min(a,c))\rangle$
$= \langle \min(\max(1-\min(a,c), \min(a,c)), \max(1-a,1-c,\min(a,c)),$
$\max(\min(a,c,\max(1-a,1-c)), \min(a,c,1-\min(a,c))))\rangle$
$= \langle \max(1-a,1-c,\min(a,c)), \min(a,c,\max(1-a,1-c))\rangle$

and

$\max(1-a,1-c,\min(a,c)) - \min(a,c,\max(1-a,1-c))$
$\geq \min(a,c) - \min(a,c,\max(1-a,1-c)) \geq 0$

The other assertions are proved analogically. $\Box$

Theorem 2.5: The following assertions are valid:

(a) If $A \Rightarrow B$ is an IFT, then $\Box A \supset \Box B$ is an IFT.
(b) If $A \Rightarrow B$ is an IFT, then $\Diamond A \supset \Diamond B$ is an IFT.
(c) If $A \Leftrightarrow B$ is an IFT, then $\Box A \equiv \Box B$ is an IFT.
(d) If $A \Leftrightarrow B$ is an IFT, then $\Diamond A \equiv \Diamond B$ is an IFT.

Proof: (a) Let $A \Rightarrow B$ is an IFS, i.e., $V(A \Rightarrow B) = \langle \max(b,c), 1-\max(b,c)\rangle$. Therefore: $\max(b,c) \geq \frac{1}{2}$. Let us assume that for

$V(\Box A \supset \Box B) = \langle \max(1-a,c), \min(a,1-c)\rangle$

it is valid: $\max(1-a,c) < \min(a,1-c)$. Then: $a > \frac{1}{2}, a > c$, $c < \frac{1}{2}$. Therefore, $b \leq 1-a < \frac{1}{2}$, i.e., $\max(b,c) < \frac{1}{2}$, which is a contradiction, i.e., $\Box A \supset \Box B$ is an IFT.

(b)-(d) are proved analogically. □

3. Remarks to the preprint IM-MFAIS-5-88 [1]

To sg-variant and (max-min)-variant of the intuitionistic fuzzy propositional calculus (IFPC) we shall add some new results.

Obviously every tautology is an IFT. Hence Theorem 2 [1] is valid about IFT and the following assertion also is valid (in the condition of this theorem there exists a mistake – A, B and C are propositional forms):

Theorem 3.1: If A, B and C are propositional forms, then

$(\neg A \supset \neg B) \supset ((\neg A \supset B) \supset A)$ is an IFT.

Proof: $(\neg A \supset \neg B) \supset ((\neg A \supset B) \supset A)$
$= \langle b,a\rangle \to \langle d,c\rangle) \to (\langle 1-(1-a).\text{sg}(b-c), d.\text{sg}(d-a)\rangle \to \langle a,b\rangle)$
$\langle 1-(1-a).\text{sg}(1-(1-a).\text{sg}(b-c)-a).\text{sg}(1-(1-a).\text{sg}(b-c)-a)-$
$(1-d).\text{sg}(b-d).b.\text{sg}(1-(1-a).\text{sg}(b-c)-a).\text{sg}(b-d.\text{sg}(b-c).\text{sg}(d-a))$
$.\text{sg}((1-a).\text{sg}(1-(1-a).\text{sg}(b-c)-a)-(1-d).\text{sg}(b-d)).\text{sg}(b.\text{sg}(1-(1-a).$
$\text{sg}(b-c)-a).\text{sg}(b-d.\text{sg}(b-c).\text{sg}(d-a)0-c.\text{sg}(b-d).\text{sg}(c-a))\rangle$ and: $\langle 1-(1-a).\text{sg}(1-$
$(1-a).\text{sg}(b-c)-a).\text{sg}(1-(1-a).\text{sg}(b-c)-a)-$
$(1-d).\text{sg}(b-d).b.\text{sg}(1-(1-a).\text{sg}(b-c)-a).\text{sg}(b-d.\text{sg}(b-c).\text{sg}(d-a))$

if $b > c$, from: $\text{sg}(1-(1-a).\text{sg}(b-c)-a) = \text{sg}(1-(1-a)-a) = 0$

$= \langle 1,0\rangle;$

if $b \leq c:$

$= 1-(1-a).\text{sg}((1-a)-(1-d).\text{sg}(b-d)) - b.\text{sg}(1-a).\text{sg}(1-a-(1-d).\text{sg}(b-d))$
$.\text{sg}(b.\text{sg}(1-a)-c.\text{sg}(b-d).\text{sg}(c-a);$

if $b > d:$

$\geq a-b.\text{sg}(1-a).\text{sg}(b.\text{sg}(1-a)-c.\text{sg}(c-a));$

if $c > a$:

$= a-b.\text{sg}(1-a).\text{sg}(b.\text{sg}(1-a)-c) \geq a-b.\text{sg}(b-c) = a \geq 0;$

if $c \leq a$:

$\geq a-b.\text{sg}(1-a).\text{sg}(b.\text{sg}((1-a)) \geq a-b.\text{sg}(1-a) \geq a-b \geq 0.$ □

Three variants of the Modus Ponens are valid for (max-min)-variant of the IFPC.

Theorem 3.2:

(a) If A and $A\&B$ are IFTs, then B is an IFT.

(b) If A and $\neg(A \supset B)$ are IFTs, then $\neg B$ is an IFT.

(c) If $(A\&(A \supset B)) \supset B$ is an IFT.

Proof: (a) Let us assume that $c > d$ and by the above conditions $a \geq b$ and $\min(a,c) \geq \max(b,d)$. Then: $d > c \geq \min(a,c) \geq \max(b,d) \geq d$, which is a contradiction, i.e., $c \geq d$. Hence, B is an IFT.

(b) Let us assume that $c > d$ and by the above conditions: $a \geq b$ and $\min(a,d) \geq \max(b,c)$. Then $d \geq \min(a,d) \geq \max(b,c) \geq c \geq d$, which is a contradiction, i.e., $c \leq d$. Hence $\neg B$ is an IFT.

(c) $V(A\&(A \supset B)) \supset B)$
$= \langle a,b\rangle\&(\langle a,b\rangle \to \langle c,d\rangle)) \to \langle c,d\rangle$
$= (\langle a,b\rangle\&\langle \max(b,c),\min(a,d)\rangle) \to \langle c,d\rangle$
$= \langle \min(a,\max(b,c)),\max(b,\min(a,d))\rangle \to \langle c,d\rangle$
$= \langle \max(b,c,\min(a,d)),\min(a,d,\max(b,c))\rangle.$

From

$$\max(b,c,\min(a,d)) \geq \min(a,d) \geq \min(a,d,\max(b,c))$$

follows that $(A\&(A \supset B)) \supset B$ is an IFT. □

Theorem 3.3: $A \supset (\neg A \supset B)$ is an IFT.

Proof: $V(A \supset (\neg A \supset B))$
$= \langle a,b\rangle \to \langle \max(a,c),\min(b,d)\rangle$
$= \langle \max(a,b,c),\min(a,b,d)\rangle.$

The validity of the assertion follows from the inequalities:

$$\max(a,b,c) \geq a \geq \min(a,b,d).$$

□

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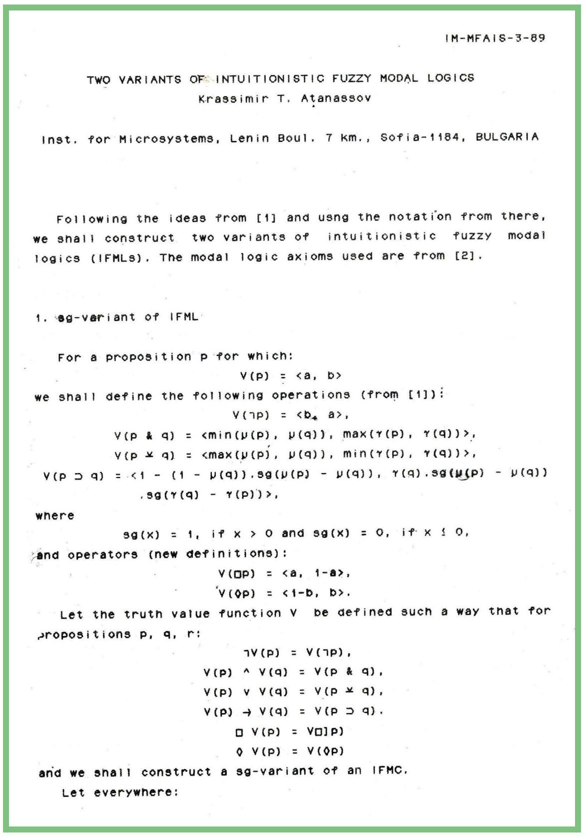

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TWO VARIANTS OF INTUITIONISTIC FUZZY MODAL LOGICS

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Following the ideas from [1] and usng the notation from there, we shall construct two variants of intuitionistic fuzzy modal logics (IFMLs). The modal logic axioms used are from [2].

1. sg-variant of IFML

For a proposition p for which:

$$V(p) = \langle a, b\rangle$$

we shall define the following operations (from [1]):

$$V(\neg p) = \langle b, a\rangle,$$

$$V(p \,\&\, q) = \langle \min(\mu(p), \mu(q)), \max(\gamma(p), \gamma(q))\rangle,$$

$$V(p \vee q) = \langle \max(\mu(p), \mu(q)), \min(\gamma(p), \gamma(q))\rangle,$$

$$V(p \supset q) = \langle 1 - (1 - \mu(q)).sg(\mu(p) - \mu(q)), \gamma(q).sg(\mu(p) - \mu(q)).sg(\gamma(q) - \gamma(p))\rangle,$$

where

$$sg(x) = 1, \text{ if } x > 0 \text{ and } sg(x) = 0, \text{ if } x \leq 0,$$

and operators (new definitions):

$$V(\Box p) = \langle a, 1-a\rangle,$$

$$V(\Diamond p) = \langle 1-b, b\rangle.$$

Let the truth value function V be defined such a way that for propositions p, q, r:

$$\neg V(p) = V(\neg p),$$

$$V(p) \wedge V(q) = V(p \,\&\, q),$$

$$V(p) \vee V(q) = V(p \vee q),$$

$$V(p) \rightarrow V(q) = V(p \supset q).$$

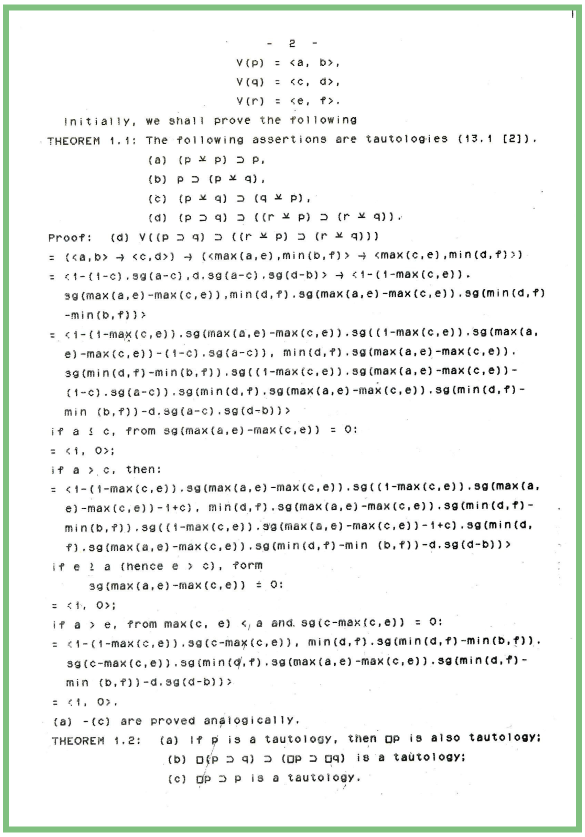
$$\Box V(p) = V\Box]p)$$

$$\Diamond V(p) = V(\Diamond p)$$

and we shall construct a sg-variant of an IFMC.

Let everywhere:

Page 1



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$$V(p) = \langle a, b\rangle,$$

$$V(q) = \langle c, d\rangle,$$

$$V(r) = \langle e, f\rangle.$$

Initially, we shall prove the following

THEOREM 1.1: The following assertions are tautologies (13.1 [2]).

(a) $(p \vee p) \supset p$,

(b) $p \supset (p \vee q)$,

(c) $(p \vee q) \supset (q \vee p)$,

(d) $(p \supset q) \supset ((r \vee p) \supset (r \vee q))$.

Proof: (d) $V((p \supset q) \supset ((r \vee p) \supset (r \vee q)))$

$= (\langle a,b\rangle \rightarrow \langle c,d\rangle) \rightarrow (\langle \max(a,e),\min(b,f)\rangle \rightarrow \langle \max(c,e),\min(d,f)\rangle)$

$= \langle 1-(1-c).sg(a-c),d.sg(a-c).sg(d-b)\rangle \rightarrow \langle 1-(1-\max(c,e)).sg(\max(a,e)-\max(c,e)),\min(d,f).sg(\max(a,e)-\max(c,e)).sg(\min(d,f)-\min(b,f))\rangle$

$= \langle 1-(1-\max(c,e)).sg(\max(a,e)-\max(c,e)).sg((1-\max(c,e)).sg(\max(a,e)-\max(c,e))-(1-c).sg(a-c)), \min(d,f).sg(\max(a,e)-\max(c,e)).sg(\min(d,f)-\min(b,f)).sg((1-\max(c,e)).sg(\max(a,e)-\max(c,e))-(1-c).sg(a-c)).sg(\min(d,f).sg(\max(a,e)-\max(c,e)).sg(\min(d,f)-\min(b,f))-d.sg(a-c).sg(d-b))\rangle$

if $a \leq c$, from $sg(\max(a,e)-\max(c,e)) = 0$:

$= \langle 1, 0\rangle$;

if $a > c$, then:

$= \langle 1-(1-\max(c,e)).sg(\max(a,e)-\max(c,e)).sg((1-\max(c,e)).sg(\max(a,e)-\max(c,e))-1+c), \min(d,f).sg(\max(a,e)-\max(c,e)).sg(\min(d,f)-\min(b,f)).sg((1-\max(c,e)).sg(\max(a,e)-\max(c,e))-1+c).sg(\min(d,f).sg(\max(a,e)-\max(c,e)).sg(\min(d,f)-\min(b,f))-d.sg(d-b))\rangle$

if $e \geq a$ (hence $e > c$), form

$sg(\max(a,e)-\max(c,e)) = 0$:

$= \langle 1, 0\rangle$;

if $a > e$, from $\max(c, e) < a$ and $sg(c-\max(c,e)) = 0$:

$= \langle 1-(1-\max(c,e)).sg(c-\max(c,e)), \min(d,f).sg(\min(d,f)-\min(b,f)).sg(c-\max(c,e)).sg(\min(d,f).sg(\max(a,e)-\max(c,e)).sg(\min(d,f)-\min(b,f))-d.sg(d-b))\rangle$

$= \langle 1, 0\rangle$.

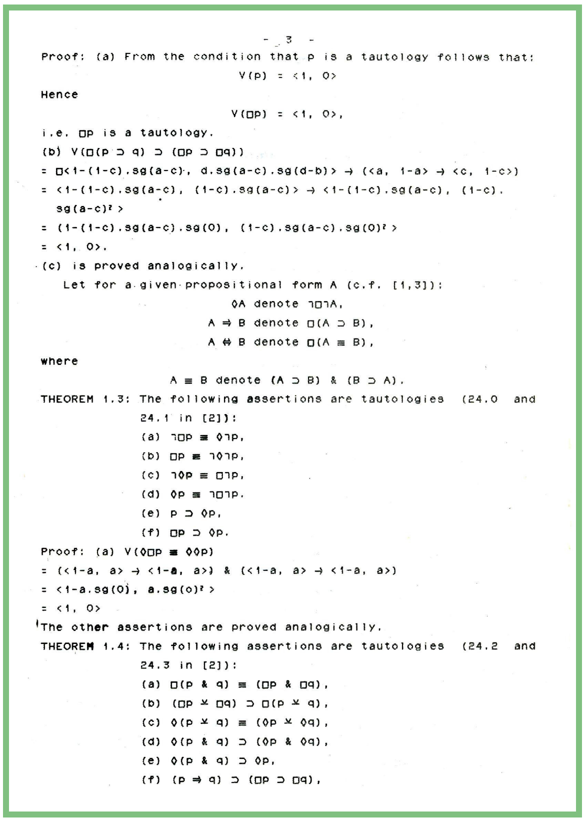
(a) -(c) are proved analogically.

THEOREM 1.2: (a) If p is a tautology, then $\Box p$ is also tautology;

(b) $\Box(p \supset q) \supset (\Box p \supset \Box q)$ is a tautology;

(c) $\Box p \supset p$ is a tautology.

Page 2



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Proof: (a) From the condition that p is a tautology follows that:

$$V(p) = \langle 1, 0\rangle$$

Hence

$$V(\Box p) = \langle 1, 0\rangle,$$

i.e. $\Box p$ is a tautology.

(b) $V(\Box(p \supset q) \supset (\Box p \supset \Box q))$

$= \Box\langle 1-(1-c).sg(a-c), d.sg(a-c).sg(d-b)\rangle \rightarrow (\langle a, 1-a\rangle \rightarrow \langle c, 1-c\rangle)$

$= \langle 1-(1-c).sg(a-c), (1-c).sg(a-c)\rangle \rightarrow \langle 1-(1-c).sg(a-c), (1-c).sg(a-c)^2\rangle$

$= (1-(1-c).sg(a-c).sg(0), (1-c).sg(a-c).sg(0)^2\rangle$

$= \langle 1, 0\rangle$.

(c) is proved analogically.

Let for a given propositional form A (c.f. [1,3]):

$$\Diamond A \text{ denote } \neg\Box\neg A,$$

$$A \Rightarrow B \text{ denote } \Box(A \supset B),$$

$$A \Leftrightarrow B \text{ denote } \Box(A \equiv B),$$

where

$$A \equiv B \text{ denote } (A \supset B) \,\&\, (B \supset A).$$

THEOREM 1.3: The following assertions are tautologies (24.0 and 24.1 in [2]):

(a) $\neg\Box p \equiv \Diamond\neg p$,

(b) $\Box p \equiv \neg\Diamond\neg p$,

(c) $\neg\Diamond p \equiv \Box\neg p$,

(d) $\Diamond p \equiv \neg\Box\neg p$.

(e) $p \supset \Diamond p$,

(f) $\Box p \supset \Diamond p$.

Proof: (a) $V(\Diamond\Box p \equiv \Diamond\Diamond p)$

$= (\langle 1-a, a\rangle \rightarrow \langle 1-a, a\rangle) \,\&\, (\langle 1-a, a\rangle \rightarrow \langle 1-a, a\rangle)$

$= \langle 1-a.sg(0), a.sg(o)^2\rangle$

$= \langle 1, 0\rangle$

The other assertions are proved analogically.

THEOREM 1.4: The following assertions are tautologies (24.2 and 24.3 in [2]):

(a) $\Box(p \,\&\, q) \equiv (\Box p \,\&\, \Box q)$,

(b) $(\Box p \vee \Box q) \supset \Box(p \vee q)$,

(c) $\Diamond(p \vee q) \equiv (\Diamond p \vee \Diamond q)$,

(d) $\Diamond(p \,\&\, q) \supset (\Diamond p \,\&\, \Diamond q)$,

(e) $\Diamond(p \,\&\, q) \supset \Diamond p$,

(f) $(p \Rightarrow q) \supset (\Box p \supset \Box q)$,

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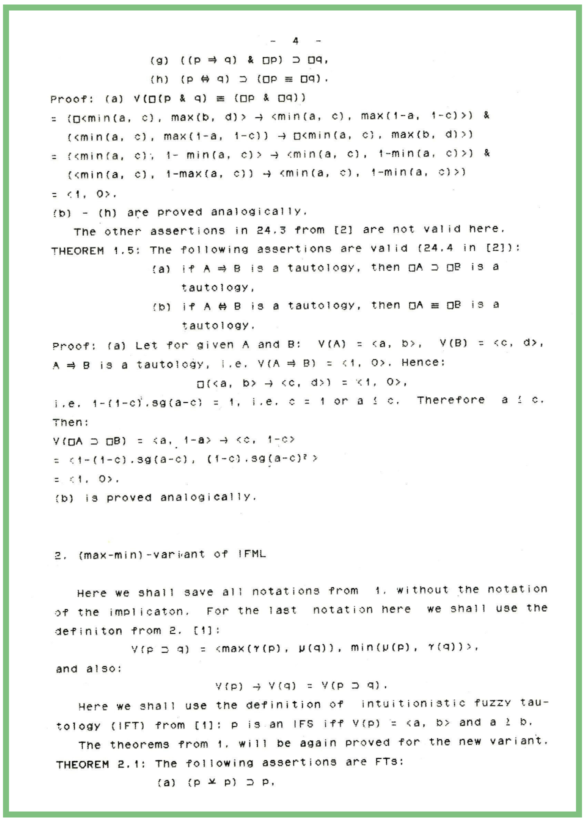

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(g) $((p \Rightarrow q) \,\&\, \Box p) \supset \Box q$,

(h) $(p \Leftrightarrow q) \supset (\Box p \equiv \Box q)$.

Proof: (a) $V(\Box(p \,\&\, q) \equiv (\Box p \,\&\, \Box q))$

$= (\Box\langle \min(a, c), \max(b, d)\rangle \rightarrow \langle \min(a, c), \max(1-a, 1-c)\rangle) \,\&\, (\langle \min(a, c), \max(1-a, 1-c)) \rightarrow \Box\langle \min(a, c), \max(b, d)\rangle)$

$= (\langle \min(a, c), 1-\min(a, c)\rangle \rightarrow \langle \min(a, c), 1-\min(a, c)\rangle) \,\&\, (\langle \min(a, c), 1-\max(a, c)) \rightarrow \langle \min(a, c), 1-\min(a, c)\rangle)$

$= \langle 1, 0\rangle$.

(b) - (h) are proved analogically.

The other assertions in 24.3 from [2] are not valid here.

THEOREM 1.5: The following assertions are valid (24.4 in [2]):

(a) if $A \Rightarrow B$ is a tautology, then $\Box A \supset \Box B$ is a tautology,

(b) if $A \Leftrightarrow B$ is a tautology, then $\Box A \equiv \Box B$ is a tautology.

Proof: (a) Let for given A and B: $V(A) = \langle a, b\rangle$, $V(B) = \langle c, d\rangle$, $A \Rightarrow B$ is a tautology, i.e. $V(A \Rightarrow B) = \langle 1, 0\rangle$. Hence:

$$\Box(\langle a, b\rangle \rightarrow \langle c, d\rangle) = \langle 1, 0\rangle,$$

i.e. $1-(1-c).sg(a-c) = 1$, i.e. $c = 1$ or $a \leq c$. Therefore $a \leq c$.

Then:

$V(\Box A \supset \Box B) = \langle a, 1-a\rangle \rightarrow \langle c, 1-c\rangle$

$= \langle 1-(1-c).sg(a-c), (1-c).sg(a-c)^2\rangle$

$= \langle 1, 0\rangle$.

(b) is proved analogically.

2. (max-min)-variant of IFML

Here we shall save all notations from 1. without the notation of the implicaton. For the last notation here we shall use the definiton from 2. [1]:

$$V(p \supset q) = \langle \max(\gamma(p), \mu(q)), \min(\mu(p), \gamma(q))\rangle,$$

and also:

$$V(p) \rightarrow V(q) = V(p \supset q).$$

Here we shall use the definition of intuitionistic fuzzy tautology (IFT) from [1]: p is an IFS iff $V(p) = \langle a, b\rangle$ and $a \geq b$.

The theorems from 1. will be again proved for the new variant.

THEOREM 2.1: The following assertions are FTs:

(a) $(p \vee p) \supset p$.

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(b) p ⊃ (p ⊻ q),
(c) (p ⊻ q) ⊃ (q ⊻ p),
(d) (p ⊃ q) ⊃ ((r ⊻ p) ⊃ (r ⊻ q)).

Proof: (d) V((p ⊃ q) ⊃ ((r ⊻ p) ⊃ (r ⊻ q)))
= (<a,b> → <c,d>) → (<max(a,e), min(b,f)> → <max(c,e), min(d,f)>)
= <max(b,d), min(a,d)> → <max(c,e,min(b,f)), min(d,f,max(a,e))>
= <max(min(a,d),c,e,min(b,f)), min(d,f,max(a,e),max(b,d))>
and
max(min(a,d),c,e,min(b,f)) - min(d,f,max(a,e),max(b,d))
≥ max(min(a,d),c,e) - min(d,max(a,e))
if a ≥ d:
= max(d,c,e) - min(d,max(a,e)) ≥ 0;
if a < d:
= max(a,c,e) - min(d,max(a,e))
≥ max(a,e) - min(d,max(a,e)) ≥ 0.
(a)- (c) are proved analogically.

THEOREM 2.2 (a) □(p ⊃ q) ⊃ (□p ⊃ □q) is an IFT,
(b) □p ⊃ p is an IFT.

Proof: (a) V(□(p ⊃ q) ⊃ (□p ⊃ □q))
= □<max(b,c), min(a,d)> ⊃ (<a,1-a> ⊃ <c,1-c>)
= <max(b,c), 1-max(b,c)> ⊃ <max(1-a,c), min(a,1-c)>
= <max(1-max(b,c),1-a,c), min(a,1-c,max(b,c))>
and
max(1-max(b,c),1-a,c) - min(a,1-c.max(b,c))
= 2.max(1-max(b,c),1-a,c) - 1
Let us assume that:
max(1-max(b,c),1-a,c) < 1/2.
Hence: max(b,c) > 1/2, 1-a < 1/2 and c < 1/2. Therefore: b > 1/2 and a > 1/2 which is a contradiction. Hence
max(1-max(b,c),1-a,c) ≥ 1/2,
with which (a) is proved.
(b) is proved directly.

THEOREM 2.3: The following assertions are IFTs:
(a) ¬□p ≡ ◊¬p,
(b) □p ≡ ¬◊¬p,
(c) ¬◊p ≡ □¬p,
(d) ◊p ≡ ¬□¬p.
(e) p ⊃ ◊p,
(f) □p ⊃ ◊p.

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Proof: (a) V(¬□p ≡ ◊¬p)
= (¬<a, 1-a> → □<b, a>) & (□<b, a> → ¬<a, 1-a>)
= (<1-a, a> → <1-a, a>) & (<1-a, a> → <1-a, a>)
= <max(a, 1-a), min(a, 1-a)>,
which is an IFS.
(b) - (f) are proved analogically.

THEOREM 2.4: The following assertions are IFTs:
(a) □(p & q) ≡ (□p & □q),
(b) (□p ⊻ □q) ⊃ □(p ⊻ q),
(c) ◊(p ⊻ q) ≡ (◊p ⊻ ◊q),
(d) ◊(p & q) ⊃ (◊p & ◊q),
(e) ◊(p & q) ⊃ ◊p,
(f) (p ⇛ q) ⊃ (□p ⊃ □q),
(g) ((p ⇛ q) & □p) ⊃ □q,
(h) (p ⇛ q) ⊃ (◊p ⊃ ◊q),
(i) ((p ⇛ q) & ◊p) ⊃ ◊q,
(j) (p ⇚⇛ q) ⊃ (□p ≡ □q).
(k) (p ⇚⇛ q) ⊃ (◊p ≡ ◊q).

Proof: (a) V(□(p & q) ≡ (□p & □q))
= (□<min(a,c), max(b,d)> → (<a,1-a> ^ <c,1-c>)) ^ ((<a,1-a> ^ <c,1-c>) → □<min(a,c), max(b,d)>)
= (<min(a,c), 1-min(a,c)> → <min(a,c), max(1-a,1-c)>) ^ (<min(a,c), max(1-a,1-c)> → <min(a,c), 1-min(a,c)>)
= <max(1-min(a,c),min(a,c)), min(a,c,max(1-a,1-c))> ^ <max(1-a,1-c,min(a,c)), min(a,c,1-min(a,c))>
= <min(max(1-min(a,c),min(a,c)), max(1-a,1-c,min(a,c))), max(min(a,c,max(1-a,1-c)), min(a,c,1-min(a,c)))>
= <max(1-a,1-c,min(a,c)), min(a,c,max(1-a.1-c))>
and
max(1-a,1-c,min(a,c)) - min(a,c,max(1-a,1-c))
≥ min(a,c) - min(a,c,max(1-a.1-c)) ≥ 0.
The other assertions are proved analogically.

THEOREM 2.5: The following assertions are valid:
(a) if A ⇛ B is an IFT, then □A ⊃ □B is an IFT.
(b) if A ⇛ B is an IFT, then ◊A ⊃ ◊B is an IFT.
(c) if A ⇚⇛ B is an IFT, then □A ≡ □B is an IFT.
(d) if A ⇚⇛ B is an IFT, then ◊A ≡ ◊B is an IFT.

Proof: (a) Let A ⇛ B is an IFS, i.e.
V(A ⇛ B) = <max(b, c), 1-max(b, c)>.

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Therefore: max(b, c) ≥ 1/2. Let us assume, that for
V(□A ⊃ □B) = <max(1-a, c), min(a, 1-c)>
it is valid: max(1-a, c) < min(a, 1-c).
Then: a > 1/2, a > c, c < 1/2. Therefore b ≤ 1-a < 1/2, i.e. max(b, c) < 1/2, which is a contradiction, i.e. □A ⊃ □B is an IFT.
(b) - (d) are proved analogically.

3. Remarks to the preprint IM-MFAIS-5-88.

To sg-variant and (max-min)-variant of the intuitionstic fuzzy propositional calculus (IFPC) we shall add some new results.

Obviously, every tautology is an IFT. Hence theorem 2 [1] is valid about IFT and the following assertion also is valid (in the condition of this theorem there exists a mistake - A, B and C are propositional forms):

THEOREM 3.1: If A, B and C are propositional forms, then
(¬A ⊃ ¬B) ⊃ ((¬A ⊃ B) ⊃ A) is a IFT.

Proof: V((¬A ⊃ ¬B) ⊃ ((¬A ⊃ B) ⊃ A))
= (<b, a>→<d, c>)→(<1-(1-a).sg(b-c), d.sg(b-c).sg(d-a)>→<a, b>)
= <1-(1-a).sg(1-(1-a).sg(b-c)-a).sg((1-a).sg(1-(1-a).sg(b-c)-a)-(1-d).sg(b-d)), b.sg(1-(1-a).sg(b-c)-a).sg(b-d.sg(b-c).sg(d-a)).sg((1-a).sg(1-(1-a).sg(b-c)-a)-(1-d).sg(b-d)).sg(b.sg(1-(1-a).sg(b-c)-a).sg(b-d.sg(b-c).sg(d-a))-c.sg(b-d).sg(c-a))> and:
1-(1-a).sg(1-(1-a).sg(b-c)-a).sg((1-a).sg(1-(1-a).sg(b-c)-a)-(1-d).sg(b-d)) - b.sg(1-(1-a).sg(b-c)-a).sg(b-d.sg(b-c).sg(d-a)).sg((1-a).sg(1-(1-a).sg(b-c)-a)-(1-d).sg(b-d)).sg(b.sg(1-(1-a).sg(b-c)-a).sg(b-d.sg(b-c),sg(d-a))-c.sg(b-d).sg(c-a))
if b > c, from: sg(1-(1-a).sg(b-c)-a) = sg(1-(1-a)-a) = 0:
= <1, 0>;
if b ≤ c:
= 1-(1-a).sg((1-a)-(1-d).sg(b-d))-b.sg(1-a).sg(1-a-(1-d).sg(b-d)).sg(b.sg(1-a)-c.sg(b-d).sg(c-a))
if b > d:
≥ a-b.sg(1-a).sg(b.sg(1-a)-c.sg(c-a))
if c > a:
= a-b.sg(1-a).sg(b.sg(1-a)-c) ≥ a-b.sg(b-c)= a ≥ 0;

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if c ≤ a:
≥ a-b.sg(1-a).sg(b.sg(1-a)) ≥ a-b.sg(1-a) ≥ a-b ≥ 0.

Three variants of the Modus Ponens are valid for (max-min)-variant of the IFPC.

THEOREM 3.2: (a) If A and (A & B) are IFTs, then B is an IFT.
(b) If A and ¬(A ⊃ B) are IFTs, then ¬B is an IFT.
(c) (A & (A ⊃ B)) ⊃ B is an IFT.

Proof: (a) Let us assume that c < d and by the above conditions are valid: a ≥ b and min(a,c) ≥ max(b,d).
Then: d > c ≥ min(a, c) ≥ max(b, d) ≥ d, which is a contradiction, i.e. c ≥ d. Hence B is an IFT.
(b) Let us assume that c > d and by the above conditions: a ≥ b and min(a,d) ≥ max(b,c). Then
d ≥ min(a, d) ≥ max(b, c) ≥ c > d
which is a contradiction, i.e. c ≤ d. Hence ◊B is an IFT.
(c) V((A & (A ⊃ B)) ⊃ B)
= (<a, b> & (<a, b> → <c, d>)) → <c, d>
= (<a, b> & <max(b, c), min(a, d)>) → <c, d>
= <min(a, max(b, c)), max(b, min(a, d))> → <c, d>
= <max(b, c, min(a, d)), min(a, d, max(b, c))>.
From
max(b, c, min(a, d)) ≥ min(a, d) ≥ min(a, d, max(b, c)) .
follows that A & (A ⊃ B)) ⊃ B is an IFT.

THEOREM 3.3: A ⊃ (¬A ⊃ B) is an IFT.

Proof: V(A ⊃ (¬A ⊃ B))
= <a, b> → <max(a, c), min(b, d)>
= <max(a, b, c), min(a, b, d)>
The validity of the assertion follows from the inequations:
max(a, b, c) ≥ a ≥ min(a, b, d)

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