On some new geometrically motivated operators over intuitionistic fuzzy sets

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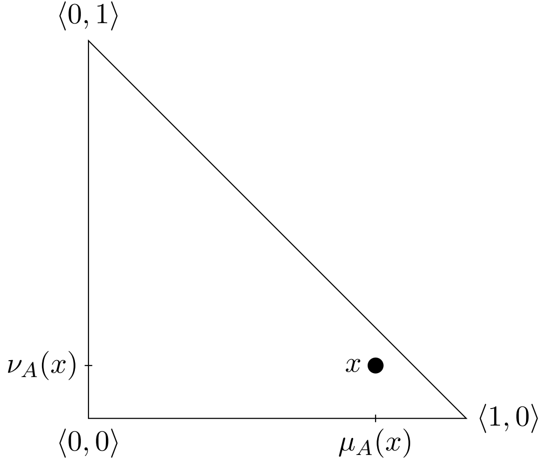
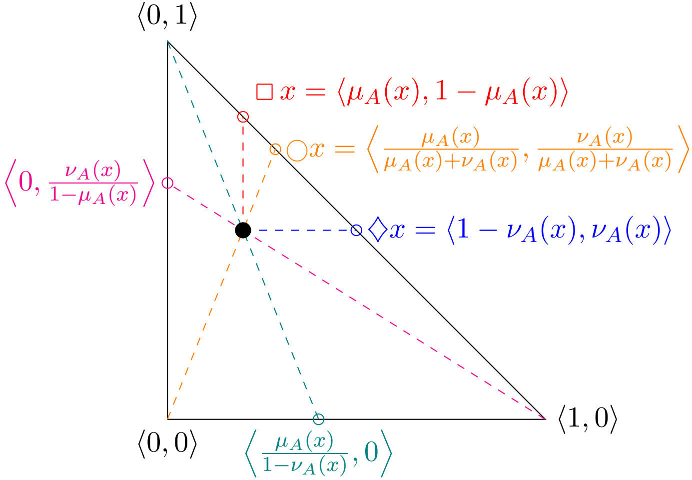
Motivation for our idea

Four new geometrically motivated operators acting in the interior of the interpretation triangle used to depict intuitionistic fuzzy sets. One uses the points obtained from the "necessity" and "possibility" operators, in combination with $\langle 0, 0 \rangle$. The second utilizes the projections of the considered point on the sides of the triangle as viewed from the points $\langle 0, 0 \rangle$, $\langle 0, 1 \rangle$, and $\langle 1, 0 \rangle$. The other two operators use two of the projections and the result of the modal operators of the internal point. We also, consider operators that are linear combination of some among the considered four, thus ending with six operators.

In what follows, we concentrate on the most used geometrical interpretation of intuitionistic fuzzy set as shown on the figure below. It represents an IFS-element x defined over the universe set X in the unitary intuitionistic fuzzy interpretation triangle.



The basis for our operators



Preliminaries

Definition (cf. Atanassov 1983)

Let E be a universe set. Let $A \subset E$. An intuitionistic fuzzy set is an object of the form

$$A^* = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\},$$

where the mappings $\mu_A : E \to [0, 1], \nu_A : E \to [0, 1]$ are such that for all $x \in E$, we have: $0 \leq \mu_A(x) + \nu_A(x) \leq 1$. The degree of membership of the element x to the set A, is denoted by $\mu_A(x)$, while the degree of non-membership of the element x to the set A is denoted by $\nu_A(x)$. The degree of indeterminacy is defined as:

$$\pi_A(x) \stackrel{\text{def}}{=} 1 - \mu_A(x) - \nu_A(x).$$

Further, for simplicity of notation we shall omit the * and simply name the IFSs by A, B, etc.
We will use also the "inclusion" between two IFSs A and B, defined over the same universe set E.

Following Atanassov 1983, we say that the IFS A is included in IFS B, and denote this fact by $A \subseteq B$ if and only if, for all $x \in E$, it is true that:

$$\begin{cases} \mu_A(x) \leq \mu_B(x) \\ \nu_A(x) \geq \nu_B(x). \end{cases}$$

Interior point

Definition

A point $\langle x, \mu_A(x), \nu_A(x)\rangle$ is said to be interior for the interpretation triangle if and only if:

$$\begin{cases} \mu_A(x).\nu_A(x) \neq 0 \\ \mu_A(x) + \nu_A(x) < 1. \end{cases} \quad (1)$$

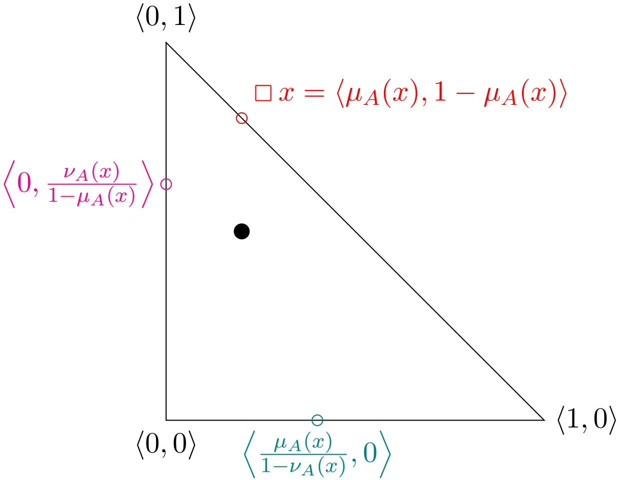
Further, for brevity, we will refer to the point $\langle x, \mu_A(x), \nu_A(x)\rangle$ by simply denoting it as x.

First operator

Definition

Let an IFS A be given. Then we can define the operator
$Z_1(A) =$

$$\begin{cases} \left\langle x, \frac{1}{3}\left(\mu_A(x) + \frac{\mu_A(x)}{1-\nu_A(x)}\right), \frac{1}{3}\left(1 - \mu_A(x) + \frac{\nu_A(x)}{1-\mu_A(x)}\right)\right\rangle & \text{if } x \text{ is an IP} \\ \langle x, \mu_A(x), \nu_A(x)\rangle & \text{otherwise }. \end{cases} \quad (2)$$

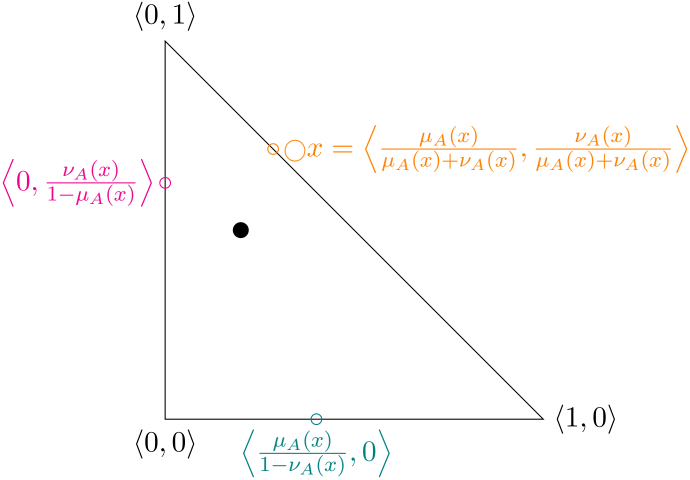

$\langle 0, 1\rangle$
$\square\, x = \langle \mu_A(x), 1 - \mu_A(x)\rangle$
$\left\langle 0, \frac{\nu_A(x)}{1-\mu_A(x)} \right\rangle$
$\langle 1, 0\rangle$
$\langle 0, 0\rangle$
$\left\langle \frac{\mu_A(x)}{1-\nu_A(x)}, 0 \right\rangle$

Second operator

Definition

Let an IFS A be given. Then we can define the operator
$Z_2(A) =$

$$\begin{cases} \left\langle x, \frac{1}{3}\left(\frac{\mu_A(x)}{\mu_A(x)+\nu_A(x)} + \frac{\mu_A(x)}{1-\nu_A(x)}\right), \frac{1}{3}\left(\frac{\nu_A(x)}{\mu_A(x)+\nu_A(x)} + \frac{\nu_A(x)}{1-\mu_A(x)}\right)\right\rangle & \text{if } x \text{ is an IP} \\ \langle x, \mu_A(x), \nu_A(x)\rangle & \text{otherwise .} \end{cases} \quad (3)$$

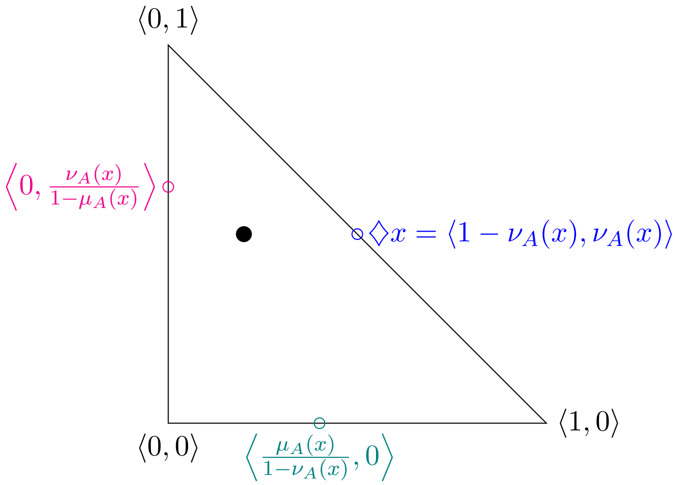

$\langle 0, 1 \rangle$
$\left\langle 0, \frac{\nu_A(x)}{1-\mu_A(x)} \right\rangle$
$\bigcirc x = \left\langle \frac{\mu_A(x)}{\mu_A(x)+\nu_A(x)}, \frac{\nu_A(x)}{\mu_A(x)+\nu_A(x)} \right\rangle$
$\langle 1, 0 \rangle$
$\langle 0, 0 \rangle$
$\left\langle \frac{\mu_A(x)}{1-\nu_A(x)}, 0 \right\rangle$

Third operator

Definition

Let an IFS A be given. Then we can define the operator $Z_3(A) =$

$$\begin{cases} \left\langle x, \frac{1}{3}\left(1 - \nu_A(x) + \frac{\mu_A(x)}{1-\nu_A(x)}\right), \frac{1}{3}\left(\nu_A(x) + \frac{\nu_A(x)}{1-\mu_A(x)}\right)\right\rangle \text{ if } x \text{ is an IP} \\ \langle x, \mu_A(x), \nu_A(x)\rangle \text{ otherwise .} \end{cases} \quad (4)$$



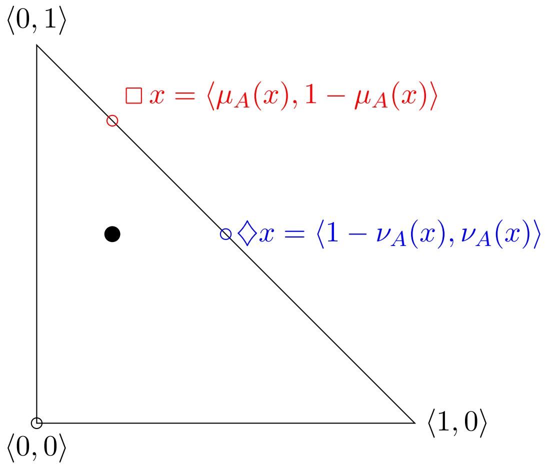
$\langle 0, 1 \rangle$
$\left\langle 0, \frac{\nu_A(x)}{1-\mu_A(x)} \right\rangle$
$\Diamond x = \langle 1 - \nu_A(x), \nu_A(x) \rangle$
$\langle 0, 0 \rangle$
$\left\langle \frac{\mu_A(x)}{1-\nu_A(x)}, 0 \right\rangle$
$\langle 1, 0 \rangle$

Fourth operator

Definition

Let an IFS A be given. Then we can define the operator $Z_4(A) =$

$$\begin{cases} \left\langle x, \frac{1}{3}\left(\mu_A(x) + 1 - \nu_A(x)\right), \frac{1}{3}\left(\nu_A(x) + 1 - \mu_A(x)\right)\right\rangle & \text{if } x \text{ is an IP} \\ \langle x, \mu_A(x), \nu_A(x)\rangle & \text{otherwise .} \end{cases} \tag{5}$$


$\langle 0, 1 \rangle$
$\square x = \langle \mu_A(x), 1 - \mu_A(x) \rangle$
$\lozenge x = \langle 1 - \nu_A(x), \nu_A(x) \rangle$
$\langle 1, 0 \rangle$
$\langle 0, 0 \rangle$

The operators Z_1, Z_2, Z_3, Z_4 are well defined.

Proof: The fact that $\mu_{Z_i}(x) \geq 0$ and $\nu_{Z_i}(x) \geq 0$ $(i = 1, 2, 3, 4)$ is obvious.
If x is not an IP for A, then it follows that

$$\max(\mu_{Z_i}(x), \nu_{Z_i}(x)) \leq \mu_{Z_i}(x) + \nu_{Z_i}(x) = \mu_A(x) + \nu_A(x) \leq 1$$

since A is an IFS. Let x be an IP for A. Then using (1) we obtain:

$$\mu_A(x) < \frac{\mu_A(x)}{1 - \nu_A(x)} < 1 \tag{6}$$

and

$$\nu_A(x) < \frac{\nu_A(x)}{1 - \mu_A(x)} < 1 \tag{7}$$

and hence, for $i = 1, 2, 3$ from (2),(3), (4), (6) and (7), we have:

$$\max(\mu_{Z_i}(x), \nu_{Z_i}(x)) < \mu_{Z_i}(x) + \nu_{Z_i}(x)$$
$$= \frac{1}{3}\left(1 + \frac{\mu_A(x)}{1-\nu_A(x)} + \frac{\nu_A(x)}{1-\mu_A(x)}\right) < 1.$$

For $i = 4$

$$\max(\mu_{Z_4}(x), \nu_{Z_4}(x)) < \mu_{Z_4}(x) + \nu_{Z_4}(x) = \frac{2}{3} < 1.$$

This completes the proof. □

Inclusion property

Theorem

For any IFS A we have:

$$Z_1(A) \subseteq Z_2(A) \subseteq Z_3(A).$$

Proof: If x is not an IP for A, we have

$$\mu_{Z_1}(x) = \mu_{Z_2}(x) = \mu_{Z_3}(x) = \mu_A(x),$$
$$\nu_{Z_1}(x) = \nu_{Z_2}(x) = \nu_{Z_3}(x) = \nu_A(x).$$

and the the result is clear. Hence, we need to consider only the IPs. From (1), we obtain:

$$\mu_{Z_1}(x) - \mu_{Z_2}(x) = \frac{1}{3}\mu_A(x)\left(\frac{\mu_A(x) + \nu_A(x) - 1}{\mu_A(x) + \nu_A(x)}\right) < 0$$

$$\mu_{Z_2}(x) - \mu_{Z_3}(x) = \frac{1}{3}\nu_A(x)\left(\frac{\mu_A(x) + \nu_A(x) - 1}{\mu_A(x) + \nu_A(x)}\right) < 0$$

With the same reasoning we obtain:

$$\nu_{Z_2}(x) - \nu_{Z_1}(x) = \frac{1}{3}\mu_A(x)\left(\frac{\mu_A(x) + \nu_A(x) - 1}{\mu_A(x) + \nu_A(x)}\right) < 0$$

$$\mu_{Z_3}(x) - \mu_{Z_2}(x) = \frac{1}{3}\nu_A(x)\left(\frac{\mu_A(x) + \nu_A(x) - 1}{\mu_A(x) + \nu_A(x)}\right) < 0$$

Hence, for any point x we have $\mu_{Z_1}(x) \leq \mu_{Z_2}(x) \leq \mu_{Z_3}(x)$ and $\nu_{Z_1}(x) \geq \nu_{Z_2}(x) \geq \nu_{Z_3}(x)$. □

Another two operators

We can construct another two operators in the following manner.

Definition

Let an IFS A be given. Let $\alpha \in (0, 1)$. Then we can define the operator

$$Z_{12}(A) = \langle x, \alpha\mu_{Z_1}(x) + (1-\alpha)\mu_{Z_2}(x), \alpha\nu_{Z_1}(x) + (1-\alpha)\nu_{Z_2}(x)\rangle \quad (8)$$

Definition

Let an IFS A be given. Let $\alpha \in (0, 1)$. Then we can define the operator

$$Z_{23}(A) = \langle x, \alpha\mu_{Z_2}(x) + (1-\alpha)\mu_{Z_3}(x), \alpha\nu_{Z_2}(x) + (1-\alpha)\nu_{Z_3}(x)\rangle \quad (9)$$

Theorem

For any IFS A we have:

$$Z_1(A) \subseteq Z_{12}(A) \subseteq Z_2(A) \subseteq Z_{23}(A) \subseteq Z_3(A)$$

Proof We will only consider the first inclusion as the remaining inequalities are verified in the same manner. From (8) we obtain:

$$\mu_{Z_{12}}(x) - \mu_{Z_1}(x) = (1-\alpha)(\mu_{Z_2}(x) - \mu_{Z_1}(x)),$$

$$\nu_{Z_{12}}(x) - \nu_{Z_1}(x) = (1-\alpha)(\nu_{Z_2}(x) - \nu_{Z_1}(x)).$$

Due to Theorem 1 and the fact that $1-\alpha > 0$, we conclude that:

$$\mu_{Z_{12}}(x) - \mu_{Z_1}(x) = \underbrace{(1-\alpha)}_{>0}\underbrace{(\mu_{Z_2}(x) - \mu_{Z_1}(x))}_{\geq 0} \geq 0$$

$$\nu_{Z_{12}}(x) - \nu_{Z_1}(x) = \underbrace{(1-\alpha)}_{>0}\underbrace{(\nu_{Z_2}(x) - \nu_{Z_1}(x))}_{\leq 0} \leq 0$$

□

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Thank you for Your attention!