Modal topological structures, index matrices and graphs through the prism of intuitionistic fuzziness

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Introduction

In this paper we describe some examples illustrating the Modal Topological Structures (MTSs) in their particular case of Intuitionistic Fuzzy MTSs (IFMTSs). These examples are related to the Intuitionistic Fuzzy Sets (IFSs), to Index Matrices (IMs) in their particular case of Intuitionistic Fuzzy IMs (IFIMs) and to Intuitionistic Fuzzy Graphs (IFGs).

Preliminaries

Short remarks on IFSs

The concept of an IFS was introduced in 1983 with the following definition.

Let us have a fixed universe E and its subset A. The set

$$A^* = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in E\},$$

where

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1$$

is called IFS and functions $\mu_A : E \to [0, 1]$ and $\nu_A : E \to [0, 1]$ represent the *degree of membership (validity, etc.)* and *non-membership (non-validity, etc.) of element $x \in E$ to a fixed set $A \subseteq E$.* Thus, we can also define function $\pi_A : E \to [0, 1]$ by means of

$$\pi(x) = 1 - \mu(x) - \nu(x)$$

and it corresponds to *the degree of indeterminacy (uncertainty, etc.).*

One of the geometrical interpretations of an element $x \in E$ in the Intuitionistic Fuzzy Interpretation Triangle (IFIT) is shown on Fig. 1.

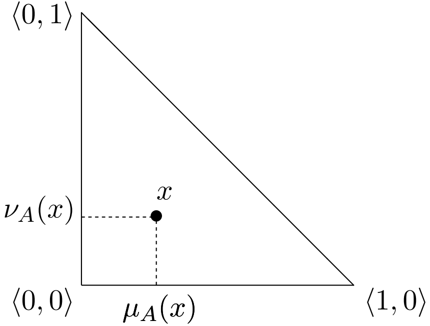


Fig. 1. Geometrical interpretation of an element $x \in E$.

A lot of operations and relations are defined over two given IFSs. Here, we use only the following of them:

$$
\begin{array}{lll}
A \subseteq B & \text{iff} & (\forall x \in E)(\mu_A(x) \leq \mu_B(x) \ \& \ \nu_A(x) \geq \nu_B(x)); \\
A \supseteq B & \text{iff} & B \subseteq A; \\
A = B & \text{iff} & (\forall x \in E)(\mu_A(x) = \mu_B(x) \ \& \ \nu_A(x) = \nu_B(x));
\end{array}
$$

$$
\begin{array}{lll}
\neg A & = & \{\langle x, \nu_A(x), \mu_A(x)\rangle \mid x \in E\}; \\
A \cap B & = & \{\langle x, \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x))\rangle | x \in E\}; \\
A \cup B & = & \{\langle x, \max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x))\rangle | x \in E\};
\end{array}
$$

A lot of operators from modal, topological and level types are defined over a given IFS. Here, we use only the two operators that represent intuitionistic fuzzy analogues of the modal operators “necessity" and “possibility":

$$\Box A = \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle \mid x \in E\},$$
$$\Diamond A = \{\langle x, 1 - \nu_A(x), \nu_A(x)\rangle \mid x \in E\},$$

and the intuitionistic fuzzy analogues of the topological operators “closure" and “interior":

$$\mathcal{C}(A) = \{\langle x, \sup_{y \in E} \mu_A(y), \inf_{y \in E} \nu_A(y)\rangle \mid x \in E\},$$
$$\mathcal{I}(A) = \{\langle x, \inf_{y \in E} \mu_A(y), \sup_{y \in E} \nu_A(y)\rangle \mid x \in E\}.$$

We must mention that for every IFS A:

$$\Box A = \neg\Diamond\neg A,$$
$$\Diamond A = \neg\Box\neg A,$$
$$\mathcal{C}(A) = \neg(\mathcal{I}(\neg A)),$$
$$\mathcal{I}(A) = \neg(\mathcal{C}(\neg A)).$$

Let everywhere below:

$$O^* = \{\langle x, 0, 1\rangle \mid x \in E\},$$

$$E^* = \{\langle x, 1, 0\rangle \mid x \in E\}.$$

When for real numbers $a, b \in [0, 1]$ it holds true that $a + b \leq 1$, the ordered pair $\langle a, b\rangle$ is called an Intuitionistic Fuzzy Pair (IFP).

Short remarks on MTSs and IFMTSs

After publishing the concept of a MTS has become an object of numerous modifications and extensions. Each of these structures was illustrated with examples from the area of the intuitionistic fuzziness. As a result, different IFMTSs were constructed.

Let us have a fixed set E and let

$$\mathcal{P}(E) = \{X | X \subseteq E\}.$$

Then,

$$\begin{aligned} \mathcal{P}(O^*) &= \{O^*\}, \\ \mathcal{P}(E^*) &= \{A | A \subseteq E^*\}, \end{aligned}$$

where A is an IFS.

Now, we will call that the operator $\mathcal{C}$ is a closure (cl)-topological operator if for each $A, B \in \mathcal{P}(E^*)$:

C1 $\mathcal{C}(A \cup B) = \mathcal{C}(A) \cup \mathcal{C}(B)$,

C2 $A \subseteq \mathcal{C}(A)$,

C3 $\mathcal{C}(\mathcal{C}(A)) = \mathcal{C}(A)$,

C4 $\mathcal{C}(O^*) = O^*$,

and that the operator $\mathcal{I}$ is an interior (in)-topological operator if for each $A, B \in \mathcal{P}$:

I1 $\mathcal{I}(A \cap B) = \mathcal{I}(A) \cap \mathcal{I}(B)$,

I2 $\mathcal{I}(A) \subseteq A$,

I3 $\mathcal{I}(\mathcal{I}(A)) = \mathcal{I}(A)$,

I4 $\mathcal{I}(E^*) = E^*$.

Now, following the definition of a topological structure and the definition of a MTS, we define that the object

$$\langle \mathcal{P}(E^*), \mathcal{O}, \zeta, \star, \eta \rangle$$

is a χ-Modal φ-Topological Structure (χ-Mφ-TS) over the set E^*, where $\mathcal{O} \in \{\mathcal{C}, \mathcal{I}\}$ is a topological operator from φ-type related with the operation $\zeta \in \{\cup, \cap\}$, and $\star \in \{\diamondsuit, \square\}$ is a modal operator from χ-type related with the operation $\eta \in \{\cup, \cap\}$, where $\varphi, \chi \in \{cl, in\}$. Therefore, each one of the both types of operators (the topological and the modal) must satisfy the respective C- or the respective I-conditions.

For clarity, we will denote the conditions related to the topological operators by Cts or Its, and these related to the modal operators by Cms or Ims, for $1 \leq s \leq 4$.
In addition, the topological and modal operators must satisfy the following additional condition (*) for each $A \in \mathcal{P}(E^*)$:

$$\star\mathcal{O}(A) = \mathcal{O}(\star A) \qquad (*)$$

that will be used again below.

Short remarks on IMs

Let I be a fixed set of indices and $\mathcal{X}$ be the set of some objects (natural, real, etc. numbers, propositions, predicates, etc.). Let operations $\circ : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ be fixed. For example, it can be “+", “.", “max", “min", or others.

Let the sets K and L satisfy the condition: $K, L \subset I$. We call an Index Matrix (IM) the object:

$$[K, L, \{a_{k_i,l_j}\}] \equiv \begin{array}{c|cccc} & l_1 & l_2 & \dots & l_n \\ \hline k_1 & a_{k_1,l_1} & a_{k_1,l_2} & \dots & a_{k_1,l_n} \\ k_2 & a_{k_2,l_1} & a_{k_2,l_2} & \dots & a_{k_2,l_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ k_m & a_{k_m,l_1} & a_{k_m,l_2} & \dots & a_{k_m,l_n} \end{array},$$

where

$$K = \{k_1, k_2, ..., k_m\} \text{ and } L = \{l_1, l_2, ..., l_n\},$$

and for $1 \le i \le m$, and for $1 \le j \le n : a_{k_i,l_j} \in \mathcal{R}$.

For the IMs $A = [K, L, \{a_{k_i,l_j}\}], B = [P, Q, \{b_{p_r,q_s}\}]$, operations that are analogous to the usual matrix operations of addition and multiplication are defined, as well as other, specific ones.

Addition

$$A \oplus_{(\circ)} B = [K \cup P, L \cup Q, \{c_{t_u,v_w}\}],$$

where

$$c_{t_u,v_w} = \begin{cases} a_{k_i,l_j}, & \text{if } t_u = k_i \in K \text{ and } v_w = l_j \in L - Q \\ & \text{or } t_u = k_i \in K - P \text{ and } v_w = l_j \in L; \\ \\ b_{p_r,q_s}, & \text{if } t_u = p_r \in P \text{ and } v_w = q_s \in Q - L \\ & \text{or } t_u = p_r \in P - K \text{ and } v_w = q_s \in Q; \\ \\ a_{k_i,l_j} \circ b_{p_r,q_s}, & \text{if } t_u = k_i = p_r \in K \cap P \\ & \text{and } v_w = l_j = q_s \in L \cap Q \\ \\ 0, & \text{otherwise} \end{cases}$$

Termwise multiplication

$$A \otimes_{(\circ)} B = [K \cap P, L \cap Q, \{c_{t_u,v_w}\}],$$

where

$$c_{t_u,v_w} = a_{k_i,l_j} \circ b_{p_r,q_s},$$

for $t_u = k_i = p_r \in K \cap P$ and $v_w = l_j = q_s \in L \cap Q$.

A lot of other operations are defined over IMs, e.g., multiplication, subtraction, etc.

When $\mathcal{X}$ is a set of IFPs, then the IM with elements from $\mathcal{X}$ is called an Intuitionistic fuzzy IM (IFIM).

Let I be a fixed set. By IFIM with index sets K and L $(K, L \subset I)$, we denote the object:

$$[K, L, \{\langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle\}]$$

$$\equiv \begin{array}{c|ccccc} & l_1 & \ldots & l_j & \ldots & l_n \\ \hline k_1 & \langle \mu_{k_1,l_1}, \nu_{k_1,l_1} \rangle & \ldots & \langle \mu_{k_1,l_j}, \nu_{k_1,l_j} \rangle & \ldots & \langle \mu_{k_1,l_n}, \nu_{k_1,l_n} \rangle \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_i & \langle \mu_{k_i,l_1}, \nu_{k_i,l_1} \rangle & \ldots & \langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle & \ldots & \langle \mu_{k_i,l_n}, \nu_{k_i,l_n} \rangle \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_m & \langle \mu_{k_m,l_1}, \nu_{k_m,l_1} \rangle & \ldots & \langle \mu_{k_m,l_j}, \nu_{k_m,l_j} \rangle & \ldots & \langle \mu_{k_m,l_n}, \nu_{k_m,l_n} \rangle \end{array},$$

where for every $1 \leq i \leq m, 1 \leq j \leq n$:
$0 \leq \mu_{k_i,l_j}, \nu_{k_i,l_j}, \mu_{k_i,l_j} + \nu_{k_i,l_j} \leq 1$ and

$$K = \{k_1, k_2, ..., k_m\}, \qquad L = \{l_1, l_2, ..., l_n\}.$$

Here, we will mention that the two above operators now for the IFIMs

$$A = [K, L, \{\langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle\}],$$

$$B = [P, Q, \{\langle \rho_{p_r,q_s}, \sigma_{p_r,q_s} \rangle\}]$$

have the following forms:

Addition-($\vee$)

or **Addition-(max,min)**

$$A \oplus_{(\vee)} B = A \oplus_{(\max,\min)} B = [K \cup P, L \cup Q, \{\langle \varphi_{t_u,v_w}, \psi_{t_u,v_w} \rangle\}],$$

where

$$\langle \varphi_{t_u,v_w}, \psi_{t_u,v_w} \rangle = \langle \mu_{k_i,l_j} \vee \rho_{p_r,q_s}, \nu_{k_i,l_j} \wedge \sigma_{p_r,q_s} \rangle$$

$$= \begin{cases} \langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle, & \text{if } t_u = k_i \in K \text{ and } v_w = l_j \in L - Q \\ & \quad \text{or } t_u = k_i \in K - P \text{ and } v_w = l_j \in L; \\ \langle \rho_{p_r,q_s}, \sigma_{p_r,q_s} \rangle, & \text{if } t_u = p_r \in P \text{ and } v_w = q_s \in Q - L \\ & \quad \text{or } t_u = p_r \in P - K \text{ and } v_w = q_s \in Q; \\ \langle \max(\mu_{k_i,l_j}, \rho_{p_r,q_s}), & \text{if } t_u = k_i = p_r \in K \cap P \\ \ \min(\nu_{k_i,l_j}, \sigma_{p_r,q_s}) \rangle, & \quad \text{and } v_w = l_j = q_s \in L \cap Q \\ \langle 0, 1 \rangle, & \text{otherwise} \end{cases}$$

Addition-($\wedge$) or **Addition-(min,max)**

$$A \oplus_{(\min,\max)} B = [K \cup P, L \cup Q, \{\langle \varphi_{t_u,v_w}, \psi_{t_u,v_w} \rangle\}],$$

where

$$\langle \varphi_{t_u,v_w}, \psi_{t_u,v_w} \rangle = \langle \mu_{k_i,l_j} \wedge \rho_{p_r,q_s}, \nu_{k_i,l_j} \vee \sigma_{p_r,q_s} \rangle$$

$$= \begin{cases} \langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle, & \text{if } t_u = k_i \in K \text{ and } v_w = l_j \in L - Q \\ & \quad \text{or } t_u = k_i \in K - P \text{ and } v_w = l_j \in L; \\ \langle \rho_{p_r,q_s}, \sigma_{p_r,q_s} \rangle, & \text{if } t_u = p_r \in P \text{ and } v_w = q_s \in Q - L \\ & \quad \text{or } t_u = p_r \in P - K \text{ and } v_w = q_s \in Q; \\ \langle \min(\mu_{k_i,l_j}, \rho_{p_r,q_s}), & \text{if } t_u = k_i = p_r \in K \cap P \\ \quad \max(\nu_{k_i,l_j}, \sigma_{p_r,q_s}) \rangle, & \quad \text{and } v_w = l_j = q_s \in L \cap Q \\ \langle 0, 1 \rangle, & \text{otherwise} \end{cases}$$

Termwise multiplication-($\vee$)

or **Termwise multiplication-(max,min)**

$$A \otimes_{(\vee)} B = A \otimes_{(\max,\min)} B = [K \cap P, L \cap Q, \{\langle \varphi_{t_u,v_w}, \psi_{t_u,v_w} \rangle\}],$$

where

$$\begin{aligned}\langle \varphi_{t_u,v_w}, \psi_{t_u,v_w} \rangle &= \langle \mu_{k_i,l_j} \vee \rho_{p_r,q_s}, \nu_{k_i,l_j} \wedge \sigma_{p_r,q_s} \rangle \\ &= \langle \max(\mu_{k_i,l_j}, \rho_{p_r,q_s}), \min(\nu_{k_i,l_j}, \sigma_{p_r,q_s}) \rangle.\end{aligned}$$

Termwise multiplication-($\wedge$) or **Termwise multiplication-(min,max)**

$$A \otimes_{(\wedge)} B = A \otimes_{(\min,\max)} B = [K \cap P, L \cap Q, \{\langle \varphi_{t_u,v_w}, \psi_{t_u,v_w} \rangle\}],$$

where

$$\begin{aligned}\langle \varphi_{t_u,v_w}, \psi_{t_u,v_w} \rangle &= \langle \mu_{k_i,l_j} \wedge \rho_{p_r,q_s}, \nu_{k_i,l_j} \vee \sigma_{p_r,q_s} \rangle \\ &= \langle \min(\mu_{k_i,l_j}, \rho_{p_r,q_s}), \max(\nu_{k_i,l_j}, \sigma_{p_r,q_s}) \rangle.\end{aligned}$$

Classical negation of an IFIM

$$\neg A = [K, L, \{\neg\langle \mu_{k_i,l_j}, \nu_{k_i,l_j}\rangle\}] = [K, L, \{\langle \nu_{k_i,l_j}, \mu_{k_i,l_j}\rangle\}],$$

A lot of other operations are defined over an extension of the IFIMs, that in the particular case are valid for the IFIM, e.g., multiplication, subtraction, etc.

Some relations are defined over IMs (that do not exist for standard matrices). Below, we will use only the following one of them:

The non-strict relation "inclusion about value" is

$$A \subseteq_v B \text{ iff } (K^* = P^*) \ \& \ (L^* = Q^*) \ \& \ (\forall k \in K)(\forall l \in L)$$

$$(\langle a_{k,l}, b_{k,l}\rangle \le \langle c_{k,l}, d_{k,l}\rangle).$$

Intuitionistic fuzzy index matrices as modal topological structures

Let us have two sets of objects Y and Z and let $\mathcal{A} \in \mathcal{P}(Y \times Z)$. Therefore, $\mathcal{A} = K \times L$ for some sets $K \subseteq Y$ and $L \subseteq Y$. In the particular case, the sets Y and Z can be subsets of the set of indices I. The elements of the set Y can be interpreted as the indices of an IM rows and the elements of the set Z can be interpreted as the indices of the columns of the same IM. If the elements of the IM are IFPs, then the set

$$A^* = \{\langle\langle k, l\rangle, \mu(k, l), \nu(k, l)\rangle \mid \langle k, l\rangle \in \mathcal{A}\},$$

where $\mu(k, l), \nu(k, l), \mu(k, l) + \nu(k, l) \in [0, 1]$ is, obviously, an IFS and

$$A^* \in \mathcal{P}(E^*) = \mathcal{P}(Y \times Z \times \mathcal{L}^*),$$

where

$$\mathcal{L}^* = \{\langle a, b\rangle \mid a, b, a + b \in [0, 1]\}.$$

Now, we can construct the IFIM

$$A = [K, L, \{\langle \mu(k,l), \nu(k,l)\rangle \mid k \in K, l \in L\}].$$

Obviously, the IFIM A is another form of the IFS A^*. Therefore, on it we can define topological and modal operators as follows:

$$\mathcal{C}(A) = [K, L, \{\langle \sup_{\langle k,l\rangle \in K\times L} \mu(k,l), \inf_{\langle k,l\rangle \in K\times L} \nu(k,l)\rangle \mid k \in K, l \in L\}],$$

$$\mathcal{I}(A) = [K, L, \{\langle \inf_{\langle k,l\rangle \in K\times L} \mu(k,l), \sup_{\langle k,l\rangle \in K\times L} \nu(k,l)\rangle \mid k \in K, l \in L\}],$$

$$\Box A = [K, L, \{\langle \mu(k,l), 1 - \mu(k,l) \mid k \in K, l \in L\}],$$

$$\Diamond A = [K, L, \{\langle 1 - \nu(k,l), \nu(k,l) \mid k \in K, l \in L\}].$$

Now, we can prove the following assertion.

Theorem 1. For every two sets Y and Z:

(1) $\langle \mathcal{P}(Y \times Z), \mathcal{C}, \oplus_\vee, \Diamond, \oplus_\vee \rangle$ is an IFMTS,

(2) $\langle \mathcal{P}(Y \times Z), \mathcal{C}, \oplus_\vee, \Diamond, \oplus_\wedge \rangle$ is an IFMTS,

(3) $\langle \mathcal{P}(Y \times Z), \mathcal{C}, \oplus_\vee, \Box, \oplus_\vee \rangle$ is an IFMTS,

(4) $\langle \mathcal{P}(Y \times Z), \mathcal{C}, \oplus_\vee, \Box, \oplus_\wedge \rangle$ is an IFMTS,

(5) $\langle \mathcal{P}(Y \times Z), \mathcal{C}, \otimes_\vee, \Diamond, \oplus_\vee \rangle$ is an IFMTS,

(6) $\langle \mathcal{P}(Y \times Z), \mathcal{C}, \otimes_\vee, \Diamond, \oplus_\wedge \rangle$ is an IFMTS,

(7) $\langle \mathcal{P}(Y \times Z), \mathcal{C}, \otimes_\vee, \Box, \oplus_\vee \rangle$ is an IFMTS,

(8) $\langle \mathcal{P}(Y \times Z), \mathcal{C}, \otimes_\vee, \Box, \oplus_\wedge \rangle$ is an IFMTS,

(9) $\langle \mathcal{P}(Y \times Z), \mathcal{I}, \oplus_\wedge, \Diamond, \oplus_\vee \rangle$ is an IFMTS,

(10) $\langle \mathcal{P}(Y \times Z), \mathcal{I}, \oplus_\wedge, \Diamond, \oplus_\wedge \rangle$ is an IFMTS,

(11) $\langle \mathcal{P}(Y \times Z), \mathcal{I}, \oplus_{\wedge}, \Box, \oplus_{\vee} \rangle$ is an IFMTS,

(12) $\langle \mathcal{P}(Y \times Z), \mathcal{I}, \oplus_{\wedge}, \Box, \oplus_{\wedge} \rangle$ is an IFMTS,

(13) $\langle \mathcal{P}(Y \times Z), \mathcal{I}, \otimes_{\wedge}, \Diamond, \oplus_{\vee} \rangle$ is an IFMTS,

(14) $\langle \mathcal{P}(Y \times Z), \mathcal{I}, \otimes_{\wedge}, \Diamond, \oplus_{\wedge} \rangle$ is an IFMTS,

(15) $\langle \mathcal{P}(Y \times Z), \mathcal{I}, \otimes_{\wedge}, \Box, \oplus_{\vee} \rangle$ is an IFMTS,

(16) $\langle \mathcal{P}(Y \times Z), \mathcal{I}, \otimes_{\wedge}, \Box, \oplus_{\wedge} \rangle$ is an IFMTS.

Proof Let the sets Y and Z be given and let

$$A = [K, L, \{\langle \mu(k,l), \nu(k,l) \rangle \mid k \in K, l \in L\}] \in Y \times Z,$$

$$B = [P, Q, \{\langle \mu(p,q), \nu(p,q) \rangle \mid p \in P, q \in Q\}] \in Y \times Z.$$

Then for example for (1) we can check the conditions that the object is an IFMTS as follows

[Ct1]

$$
\begin{aligned}
&\mathcal{C}(A \oplus_\vee B)\\
&= \mathcal{C}([K \cup P, L \cup Q, \{\langle \mu(k,l) \vee \mu(p,q), \nu(k,l) \wedge \nu(p,q)\rangle \mid\\
&\qquad\qquad k, p \in K \cup P; l, q \in L \cup Q\}])\\
&= [K \cup P, L \cup Q, \{\langle \sup_{K\cup P, L\cup Q} (\mu(k,l) \vee \mu(p,q)),\\
&\qquad \inf_{K\cup P, L\cup Q} (\nu(k,l) \wedge \nu(p,q))\rangle \mid k, p \in K \cup P; l, q \in L \cup Q\}]\\
&= [K \cup P, L \cup Q, \{\langle \sup_{K\cup P, L\cup Q} \mu(k,l) \vee \sup_{K\cup P, L\cup Q} \mu(p,q),\\
&\qquad \inf_{K\cup P, L\cup Q} \nu(k,l) \wedge \inf_{K\cup P, L\cup Q} \nu(p,q))\rangle \mid k \in K, l \in L\}]\\
&= [K, L, \{\langle \sup_{K,L} \mu(k,l), \inf_{K,L} \nu(k,l)\rangle \mid k \in K, l \in L\}]\\
&\quad \oplus_\vee [P, Q, \{\langle \sup_{P,Q} \mu(p,q), \inf_{P,Q} \nu(p,q))\rangle \mid p \in P, q \in Q\}]\\
&= \mathcal{C}(A) \oplus_\vee \mathcal{C}(B);
\end{aligned}
$$

[Ct2]

$$\begin{aligned}
\mathcal{C}(\mathcal{C}(A)) &= \mathcal{C}([K, L, \{\langle \sup_{\langle k,l\rangle \in K\times L} \mu(k,l), \inf_{\langle k,l\rangle \in K\times L} \nu(k,l)\rangle \mid k \in K, l \in L \\
&= [K, L, \{\langle \sup_{\langle p,q\rangle \in K\times L} \sup_{\langle k,l\rangle \in K\times L} \mu(k,l), \\
&\qquad \inf_{\langle p,q\rangle \in K\times L} \inf_{\langle k,l\rangle \in K\times L} \nu(k,l)\rangle \mid k \in K, l \in L\}] \\
&= [K, L, \{\langle \sup_{\langle k,l\rangle \in K\times L} \mu(k,l), \inf_{\langle k,l\rangle \in K\times L} \nu(k,l)\rangle \mid k \in K, l \in L\}] \\
&= \mathcal{C}(A);
\end{aligned}$$

[Ct3]

$$\begin{aligned}
A &= [K, L, \{\langle \mu(k,l), \nu(k,l)\rangle \mid k \in K, l \in L\}] \\
&\subseteq_v [K, L, \{\langle \sup_{\langle k,l\rangle \in K\times L} \mu(k,l), \inf_{\langle k,l\rangle \in K\times L} \nu(k,l)\rangle \mid k \in K, l \in L\}]) \\
&= \mathcal{C}(A);
\end{aligned}$$

[Ct4]

$$\begin{aligned}\mathcal{C}(O^*) &= [K, L, \{\langle \sup_{\langle k,l\rangle \in K\times L} 0, \inf_{\langle k,l\rangle \in K\times L} 1\rangle \mid k \in K, l \in L\}] \\ &= [K, L, \{\langle 0, 1\rangle \mid k \in K, l \in L\}] \\ &= O^*;\end{aligned}$$

[Cm1]

$$\begin{aligned}&\Diamond(A \oplus_{\vee} B) \\ &= \Diamond([K \cup P, L \cup Q, \{\langle \mu(k,l) \vee \mu(p,q), \nu(k,l) \wedge \nu(p,q)\rangle \mid \\ &\qquad k, p \in K \cup P; l, q \in L \cup Q\}]) \\ &= [K \cup P, L \cup Q, \{\langle 1 - (\nu(k,l) \wedge \nu(p,q)), \nu(k,l) \wedge \nu(p,q))\rangle \mid \\ &\qquad k, p \in K \cup P; l, q \in L \cup Q\}] \\ &= [K \cup P, L \cup Q, \{\langle (1 - \nu(k,l)) \vee (1 - \nu(p,q)), \nu(k,l) \wedge \nu(p,q))\rangle \mid \\ &\qquad k \in K, l \in L\}] \\ &= [K, L, \{\langle 1 - \nu(k,l), \nu(k,l)\rangle \mid k \in K, l \in L\}]\end{aligned}$$

[Cm2]

$$\begin{aligned}\Diamond\Diamond A &= \Diamond[K, L, \{\langle 1-\nu(k,l), \nu(k,l)\rangle \mid k \in K, l \in L\}]) \\ &= [K, L, \{\langle 1-\nu(k,l), \nu(k,l)\rangle \mid k \in K, l \in L\}]) \\ &= \Diamond A;\end{aligned}$$

[Cm3]

$$\begin{aligned}A &= [K, L, \{\langle \mu(k,l), \nu(k,l)\rangle \mid k \in K, l \in L\}] \\ &\subseteq_v [K, L, \{\langle 1-\nu(k,l), \nu(k,l)\rangle \mid k \in K, l \in L\}]) \\ &= \Diamond A.\end{aligned}$$

[Cm4]

$$\begin{aligned}\Diamond E^* &= \Diamond[K, L, \{\langle 1, 0\rangle \mid k \in K, l \in L\}] \\ &= [K, L, \{\langle 1, 0\rangle \mid k \in K, l \in L\}] \\ &= E^*.\end{aligned}$$

[(*)]

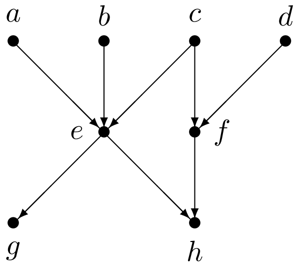
$$\begin{aligned}
\mathcal{C}(\diamondsuit A)) &= \mathcal{C}([K, L, 1-\nu(k,l), \nu(k,l)\rangle \mid k \in K, l \in L\}]) \\
&= [K, L, \{\langle \sup_{\langle k,l\rangle \in K\times L} (1-\nu(k,l)), \\
&\qquad \inf_{\langle k,l\rangle \in K\times L} \nu(k,l)\rangle \mid k \in K, l \in L\}]) \\
&= [K, L, \{\langle 1 - \inf_{\langle k,l\rangle \in K\times L} (1-\nu(k,l)), \\
&\qquad \inf_{\langle k,l\rangle \in K\times L} \nu(k,l)\rangle \mid k \in K, l \in L\}]) \\
&= \diamondsuit \mathcal{C}(A).
\end{aligned}$$

The remaining 15 assertions are proved in the same manner. □

From these examples it is clear that while topological operators determine the maximum and minimum values of the objects of K and L, modal operators determine the maximum and minimum value of any distinct pair of elements of $K \times L$.

Intuitionistic fuzzy graphs as IFMTSs

Let us have the following oriented graph C



For it, we can construct an IM with elements from the set $\{0, 1\}$, which is an adjacency matrix of the graph

$$C = \begin{array}{c|cccccccc} & a & b & c & d & e & f & g & h \\ \hline a & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ b & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ c & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ d & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ e & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ f & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ g & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ h & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}.$$

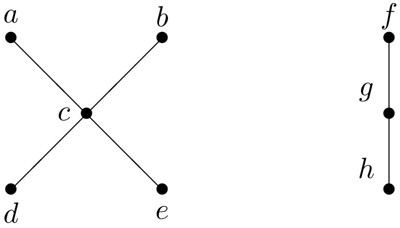
Obviously, the IM in this form is more complex than the standard (adjacency) matrix representing this graph, because both matrices have 64 symbols 0 or 1, but the IM has additional 16 alphabetic symbols – the indices of its rows and columns. But having in mind that the columns indexed by a, b, c, d and the rows, indexed by g, h contain only zeros and do not give any important information, we can transform the adjacency IM to the form

$$D = \begin{array}{c|cccc} & e & f & g & h \\ \hline a & 1 & 0 & 0 & 0 \\ b & 1 & 0 & 0 & 0 \\ c & 1 & 0 & 0 & 0 \\ d & 0 & 1 & 1 & 0 \\ e & 0 & 0 & 1 & 1 \\ f & 0 & 0 & 0 & 1 \end{array},$$

in which the initial and final vertices are omitted. This new $(0, 1)$-IM can be called “reduced adjacency IM".

Obviously, it contains only 34 symbols (24 symbols for 0 or 1 and 10 symbols for letters). Therefore the use of IMs for representation of graph is useful. In previous research of the author, it is shown that the IM representation of the graphs keep all possibilities of standard matrix representation, but it also provides a number of additional advantages. So, below we will use IMs.

Obviouly, this reduction is impossible for non-oriented graphs, but if we have a graph with two or more compoments, then the IM-representation again will be better. For example, if we have the two-component graph



it again will have a standard adjacency matrix with dimension 8×8, while its adjacency IM-representation will be

$$\begin{array}{c|ccc} & c & d & e \\ \hline a & 1 & 0 & 0 \\ b & 1 & 0 & 0 \\ c & 0 & 1 & 1 \end{array} \oplus_{(\circ)} \begin{array}{c|cc} & g & h \\ \hline f & 1 & 0 \\ g & 0 & 1 \end{array},$$

i.e., in the first case the matrix will have again 64 elements (digits 0 or 1), while in the second one – only 23 elements (13 = 9 + 4 digits and 10 = 6 + 4 letters).

Therefore, when we have a set of vertices that can be interpreted as indices of IMs, the the set of all graphs with some of these vertices will generate a topological structure.

When degrees of existing and of non-existing in the form of IFPs are given for the arcs, we obtain an IFG, which was originally introduced in 1994. Formally, it has the form

$$G = \{\langle\langle x, y\rangle, \mu_G(x, y), \nu_G(x, y)\rangle \ \langle x, y\rangle \in E_1 \times E_2\},$$

where E_1 and E_2 be two sets of vertices that are indices of the respective IFIM.

As we saw above, if the graph is an oriented one, then it is possible that $E_1 \neq E_2$, while if the graph is not oriented, then $E_1 = E_2$, but if it has more than one component, $E_1 = E_2$ will be the union of vertices sets with empty intersections.
Now, there are a lot of research over IFGs and a lot of their applications.
For the set $\mathcal{G}(V)$ of all IFGs with a fixed set of vertices V we can prove similarly to Theorem 1 that the objects

$$\langle \mathcal{P}(\mathcal{G}(V)), \mathcal{O}, \Delta, \star, \Delta \rangle,$$

where $\mathcal{O} \in \{\mathcal{C}, \mathcal{I}\}, \star \in \{\Diamond, \Box\}, \Delta \in \{\oplus_{(\circ)}, \otimes_{(\circ)}\}$ are IFMTSs.
More ptrecisely, if $V = V_I \cup V^* \cup V_O$, where V_I, V^*, V_O are, respective, the sets of input, inside and output vertices, this theorem has the form

Theorem 2.

For every two set V:

(1) $\langle \mathcal{P}((V_I \cup V^*)\times)V^* \cup V_O)), \mathcal{C}, \oplus_\vee, \Diamond, \oplus_\vee \rangle$ is an IFMTS,

(2) $\langle \mathcal{P}((V_I \cup V^*)\times)V^* \cup V_O)), \mathcal{C}, \oplus_\vee, \Diamond, \oplus_\wedge \rangle$ is an IFMTS,

(3) $\langle \mathcal{P}((V_I \cup V^*)\times)V^* \cup V_O)), \mathcal{C}, \oplus_\vee, \Box, \oplus_\vee \rangle$ is an IFMTS,

(4) $\langle \mathcal{P}((V_I \cup V^*)\times)V^* \cup V_O)), \mathcal{C}, \oplus_\vee, \Box, \oplus_\wedge \rangle$ is an IFMTS,

(5) $\langle \mathcal{P}((V_I \cup V^*)\times)V^* \cup V_O)), \mathcal{C}, \otimes_\vee, \Diamond, \oplus_\vee \rangle$ is an IFMTS,

(6) $\langle \mathcal{P}((V_I \cup V^*)\times)V^* \cup V_O)), \mathcal{C}, \otimes_\vee, \Diamond, \oplus_\wedge \rangle$ is an IFMTS,

(7) $\langle \mathcal{P}((V_I \cup V^*)\times)V^* \cup V_O)), \mathcal{C}, \otimes_\vee, \Box, \oplus_\vee \rangle$ is an IFMTS,

(8) $\langle \mathcal{P}((V_I \cup V^*)\times)V^* \cup V_O)), \mathcal{C}, \otimes_\vee, \Box, \oplus_\wedge \rangle$ is an IFMTS,

(9) $\langle \mathcal{P}((V_I \cup V^*)\times)V^* \cup V_O)), \mathcal{I}, \oplus_\wedge, \Diamond, \oplus_\vee \rangle$ is an IFMTS,

(10) $\langle \mathcal{P}((V_I \cup V^*)\times)V^* \cup V_O)), \mathcal{I}, \oplus_\wedge, \Diamond, \oplus_\wedge \rangle$ is an IFMTS,

(11) $\langle \mathcal{P}((V_I \cup V^*) \times) V^* \cup V_O)), \mathcal{I}, \oplus_\wedge, \square, \oplus_\vee \rangle$ is an IFMTS,

(12) $\langle \mathcal{P}((V_I \cup V^*) \times) V^* \cup V_O)), \mathcal{I}, \oplus_\wedge, \square, \oplus_\wedge \rangle$ is an IFMTS,

(13) $\langle \mathcal{P}((V_I \cup V^*) \times) V^* \cup V_O)), \mathcal{I}, \otimes_\wedge, \diamondsuit, \oplus_\vee \rangle$ is an IFMTS,

(14) $\langle \mathcal{P}((V_I \cup V^*) \times) V^* \cup V_O)), \mathcal{I}, \otimes_\wedge, \diamondsuit, \oplus_\wedge \rangle$ is an IFMTS,

(15) $\langle \mathcal{P}((V_I \cup V^*) \times) V^* \cup V_O)), \mathcal{I}, \otimes_\wedge, \square, \oplus_\vee \rangle$ is an IFMTS,

(16) $\langle \mathcal{P}((V_I \cup V^*) \times) V^* \cup V_O)), \mathcal{I}, \otimes_\wedge, \square, \oplus_\wedge \rangle$ is an IFMTS.

The proof is similar to the proof of Theorem 1.

Conclusion

We will finish the present paper with short remarks for a future research.
When the concept of an Extended IFIM (EIFIM) was introduced, we obtained the possibility to represent by EIFIM the graphs whose sets of vertices are IFSs of the form

$$V^* = \{\langle v, \varphi(v), \psi(v)\rangle \mid v \in V\},$$

where $\varphi(v), \psi(v), \varphi(v) + \psi(v) \in [0, 1]$, and $\varphi(v), \psi(v)$ are the degrees of existing and of non-existing of the vertices.
When the set V^* is fixed, we again can construct IFMTSs, but they will be essentially more complex. They will be objects of the next research. In this future research, we will show that these structures can be bi- or multi-MTSs.

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Thank you for attention!