# On a new expanding modal-like operator on intuitionistic fuzzy sets

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# Preliminaries

Intuitionistic fuzzy sets (IFSs) introduced in 1983 by K. Atanassov extended fuzzy sets by adding a non-membership degree $\nu_A(x)$ which reflects the extent to which an element does not belong to the set. The complement of the sum of the membership and non-membership degrees to 1 ($\pi_A(x)$) is called *hesitancy degree* or *index of indeterminacy*. A formal definition is the following:

**Definition (Atanassov, 1983)**

Let $X$ be a universe set, $A \subset X$. Then an intuitionistic fuzzy set generated by the set $A$ is an object of the form:

$$A^* = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in X\} \tag{1}$$

where $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ are mappings, such that for any $x \in X$,

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1. \tag{2}$$

# Motivation for defining the operator $T_\lambda$

Our point of departure in this investigation are the operators $G_{\alpha,\beta}$ and $F_{\alpha,\beta}$. First let us take a look at the operator $G_{\alpha,\beta}$.

**Definition (Atanassov, 1999)**

The operator $G_{\alpha,\beta}(A^*)$ for $\alpha, \beta \in [0,1]$ is defined by

$$G_{\alpha,\beta}(A^*) = \{\langle x, \alpha\mu_A(x), \beta\nu_A(x)\rangle | x \in X\} \tag{3}$$

One can easily see, that in a sense this is a contracting operator as the consecutive sequence of $(\mu(x), \nu(x))$ arising from subsequent application of the result of the operator gradually tend to $(0,0)$.

# Motivation for defining the operator $T_\lambda$

Let us now consider the operator $F_{\alpha,\beta}$.

**Definition (Atanassov, 1999)**

The operator $F_{\alpha,\beta}(A^*)$ for $\alpha, \beta, \alpha + \beta \in [0, 1]$ is defined by

$$F_{\alpha,\beta}(A^*) = \{\langle x, \mu_A(x) + \alpha\pi_A(x), \nu_A(x) + \beta\pi_A(x)\rangle | x \in X\} \quad (4)$$

One can easily see, that in a sense this is an expanding operator as the consecutive sequence of points $(\mu(x), \nu(x))$ gradually tend to a point $(a, 1 - a)$, i.e. to a fuzzy set.
The main difference in the two operators is that $G$ has multiplicative nature, while $F$ has an additive nature (in the way they treat the membership and non-membership degrees).

# Motivation for defining the operator $T_\lambda$ and Main Results

A natural question that arises is the following:
**Can we define an operator analogous to operator $F$ in the sense that it tends to fuzzy set but is of multiplicative nature as $G$?**
The answer is positive:

**Definition**

The operator $T_\lambda(A^*)$ for $\lambda \geq 0$ is defined by

$$T_\lambda(A^*) = \{\langle x, \mu_A(x) z_{T_\lambda}(x), \nu_A(x) z_{T_\lambda}(x)\rangle | x \in X\}, \qquad (5)$$

where $z_{T_\lambda}(x) = (1 - \mu_A(x))^{1+\lambda} + (1 - \nu_A(x))^{1+\lambda}$.

Let us assume that $\mu(x)\nu(x) > 0$ and $\mu(x) + \nu(x) < 1$. Then for $\lambda = 0$ from (2), and hence for $\lambda$ sufficiently small, $z_{T_\lambda}(x) > 1$, and thus the sequences of consecutive $\mu(x)$ and $\nu(x)$ are both increasing.

Let us prove that the our operator is correctly defined.

**Theorem**

*The operator $T_\lambda(A^*)$ for $\lambda \geq 0$ always produces an IFS.*

**Proof.**

Further we will make use of the fact that for any non-negative two numbers $a, b$, we have $\max(a, b) \leq a + b$. Obviously,

$$\mu T_\lambda(A^*)(x) \geq 0 \text{ and } \nu T_\lambda(A^*)(x) \geq 0.$$

Also,

$$\mu_{T_{\lambda>0}(A^*)}(x) \leq \mu_{T_0(A^*)}(x), \text{ and } \nu_{T_{\lambda>0}(A^*)}(x) \leq \nu_{T_0(A^*)}(x).$$

Thus it suffices to show that $\max(\mu_{T_0(A^*)}(x), \nu_{T_0(A^*)}(x)) \leq 1$.
A direct check shows that

$$\mu_{T_0(A^*)}(x)+\nu_{T_0(A^*)}(x) = (1-\pi_A(x))(1+\pi_A(x)) = 1-\pi_A(x)^2 \leq 1.$$

Hence, $\max(\mu_{T_0(A^*)}(x), \nu_{T_0(A^*)}(x)) \leq 1$, and (2) is fulfilled. □

# Extending $T_\lambda$

Another question that arises naturally is can we in some sense extend $T_\lambda$?
We give the following operator $T_{\lambda,\alpha,\beta}$

**Definition**

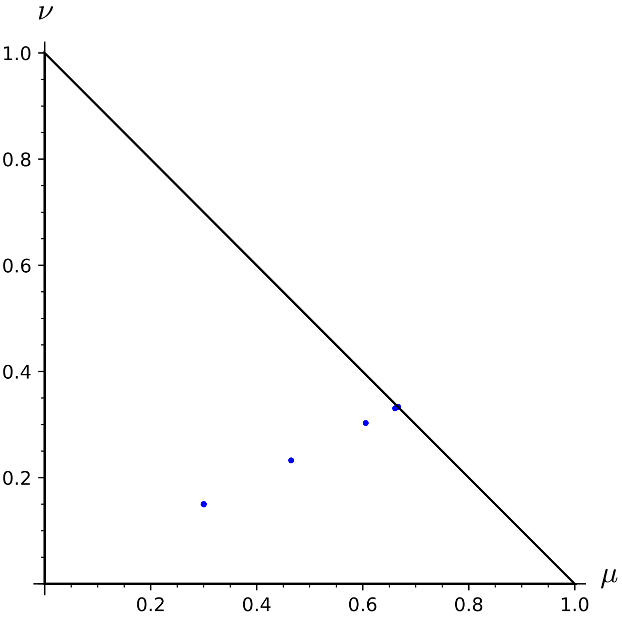
The operator $T_{\lambda,\alpha,\beta}(A^*)$ for $\lambda \geq 0$ and $\alpha, \beta \in [0,1]$ is defined by

$$T_{\lambda,\alpha,\beta}(A^*) = \{\langle x, \alpha\mu_A(x) z_{T_\lambda}(x), \beta\nu_A(x) z_{T_\lambda}(x)\rangle | x \in X\}, \qquad (6)$$

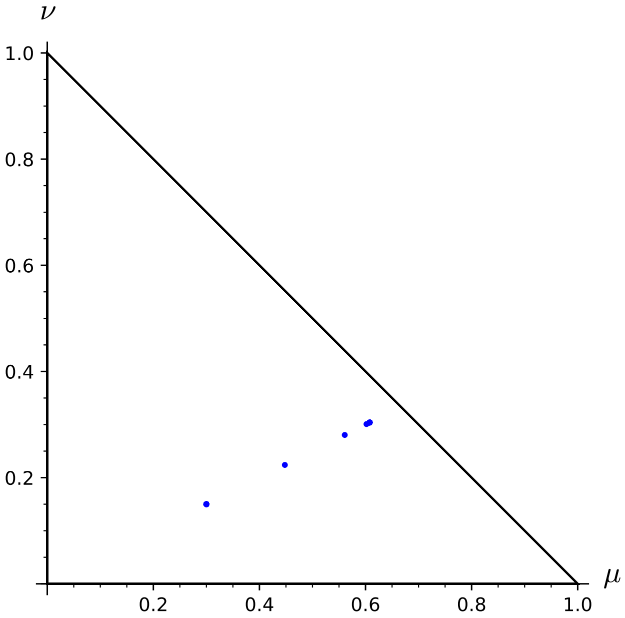
where $z_{T_\lambda}(x) = (1 - \mu_A(x))^{1+\lambda} + (1 - \nu_A(x))^{1+\lambda}$.

The proof of correctness may be done by analogy with the previous proof.
Here, we have two components fighting with each other - $z_{T_\lambda}(x)$ which pushes the point away from $(0,0)$ towards the segment $(0,1)$–$(1,0)$ and the contracting constants $\alpha, \beta$ which compact towards the abscissa or the ordinate, or to $(0,0)$.

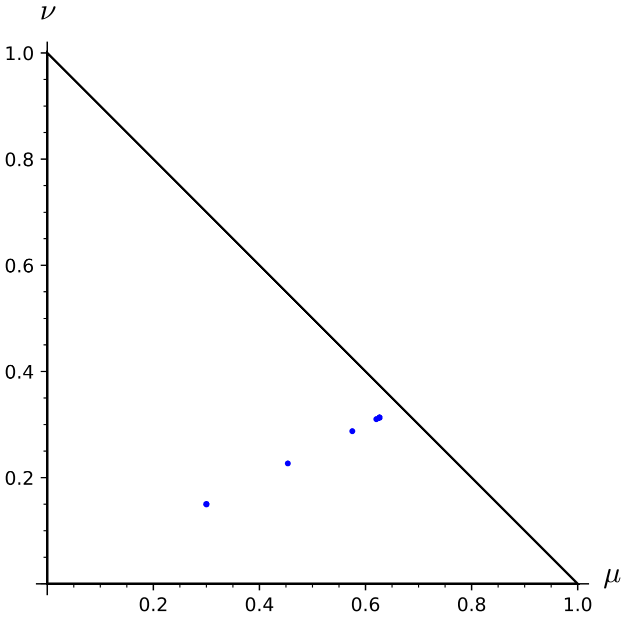
Result of application of $T_0$ to the IFS $\{\langle x, 0.3, 0.15\rangle\}$ 6 times.



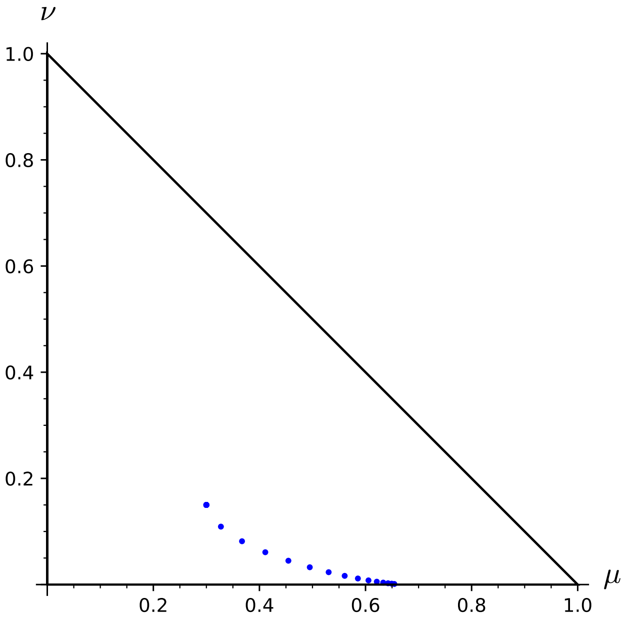
Result of application of $T_{0.15}$ to the IFS $\{\langle x, 0.3, 0.15\rangle\}$ 14 times.



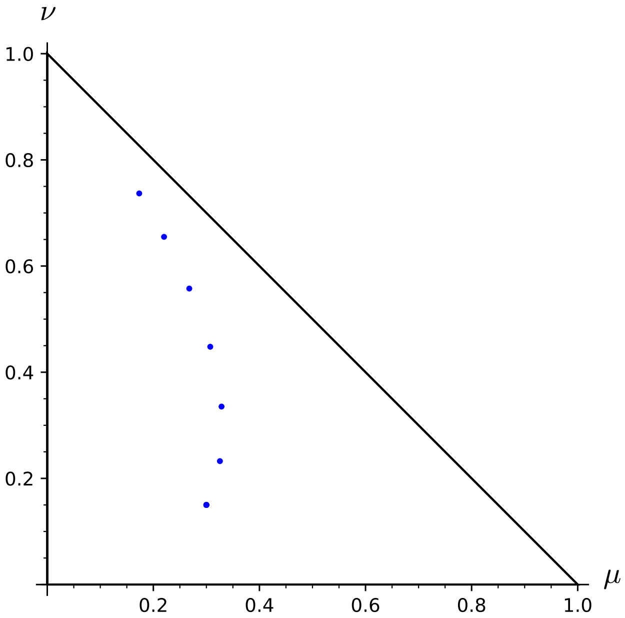
Result of application of $T_{0.1}$ to the IFS $\{\langle x, 0.3, 0.15\rangle\}$ 14 times.



## Application of $T_{1,0.9,0.6}$ to the IFS $\{\langle x, 0.3, 0.15\rangle\}$ 14 times.



Application of $T_{0,0.7,1}$ to the IFS $\{\langle x, 0.3, 0.15\rangle\}$ 14 times.



**Some final comments:** For any intuitionistic fuzzy point $\langle a, b \rangle$, such that $a + b < 1$, we can find a point $\langle a^*, b^* \rangle$, such that $T_0(\langle a^*, b^* \rangle) = \langle a, b \rangle$.
First let $a = 0$. Then we must have:

$$\begin{cases} a^*(2 - a^* - b^*) = 0 \\ b^*(2 - a^* - b^*) = b \end{cases}$$

This is only possible for $a^* = 0$. Solving further we obtain:

$$b^* = 1 - \sqrt{1 - b}$$

Further, we will assume both $a, b > 0$.

$$\begin{cases} a^*(2 - a^* - b^*) = a \\ b^*(2 - a^* - b^*) = b \end{cases}$$

Without loss of generality let us assume $\max(a, b) = a$. Then $a = tb$ for some $t = \frac{a}{b} > 1$, i.e.

$$\begin{cases} a^*(2 - a^* - b^*) = tb \\ b^*(2 - a^* - b^*) = b \end{cases}$$

Evidently, the same proportion must be present in the LHS, i.e.

$$\begin{cases} tb^*(2 - tb^* - b^*) = tb \\ b^*(2 - tb^* - b^*) = b \end{cases}$$

Solving it, after substituting back we finally obtain:

$$\begin{cases} a^* = \dfrac{a}{a+b}(1 - \sqrt{1 - a - b}) \\ b^* = \dfrac{b}{a+b}(1 - \sqrt{1 - a - b}) \end{cases}$$

# Concluding remarks

From all the above it is evident, that for the operator $T_{\lambda,\alpha,\beta}$ we can also find at least one point (or perhaps an entire set of points) that is mapped by this operator to a given intuitionistic fuzzy point $\langle a, b\rangle$, with an appropriate choice of values for $\lambda, \alpha, \beta$.
However, be that as it may, due to the limited time, our current investigation must stop here. In the future, we will study in more detail the properties of the proposed operator(s).

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Thank You for Your Attention!