Intuitionistic Fuzzy Multi-Dimensional Modal Topological Structures

Krassimir Atanassov

Dept. of Bioinformatics and Mathematical Modelling

Institute of Biophysics and Biomedical Engineering

Bulgarian Academy of Sciences

105 Acad. G. Bonchev Str., 1113 Sofia, Bulgaria

emails: krat@bas.bg, k.t.atanassvo@gmail.com

Introduction

In the middle of 2022, for a first time the concept of a Modal Topological Structure (MTS) was introduced. During the last three years, the author published a series of papers over MTSs illustrating them with different examples from the area of the intuitionistic fuzziness.

The concept of a MTS was extended to temporal MTS, level MTS, bi-MTS, multi-MTS and others structures.

Now, a new direction for extension of the concept of a MTS, is introduced and illustrated again with examples from the area of the IFSs. In the Conclusion, a geometrical interpretation of the new structures is given and some open problems are formulated.

Short remarks on IFSs

Let the (crisp) set E be fixed and let $A \subset E$ be a fixed set. We define: an IFS A^* in E is an object of the form

$$A^* = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\},$$

where functions $\mu_A : E \to [0,1]$ and $\nu_A : E \to [0,1]$ define the *degree of membership* and the *degree of non-membership* of the element $x \in E$ to the set A, respectively, and for every $x \in E$

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1.$$

Below, we will write A instead of A^*.

Over two IFSs a lot of operations and relations are defined, but here, we will recall only the following:

$A \subseteq B$ if and only if $(\forall x \in E)(\mu_A(x) \leq \mu_B(x) \; \& \; \nu_A(x) \geq \nu_B(x))$;

$A \supseteq B$ if and only if $B \subseteq A$;

$A = B$ if and only if $(\forall x \in E)(\mu_A(x) = \mu_B(x) \; \& \; \nu_A(x) = \nu_B(x))$;

$A \cap B = \{\langle x, \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x))\rangle \mid x \in E\}$;

$A \cup B = \{\langle x, \max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x))\rangle \mid x \in E\}$;

$\Box A = \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle | x \in E\}$;

$\Diamond A = \{\langle x, 1 - \nu_A(x), \nu_A(x)\rangle \mid x \in E\}$;

$\mathcal{C}(A) = \{\langle x, \sup_{y \in E} \mu_A(y), \inf_{y \in E} \nu_A(y)\rangle \mid x \in E\}$;

$\mathcal{I}(A) = \{\langle x, \inf_{y \in E} \mu_A(y), \sup_{y \in E} \nu_A(y)\rangle \mid x \in E\}$.

Let everywhere below

$$\mathcal{P}(X) = \{Y | Y \subseteq X\},$$

where X is an arbitrary set, and

$$O^* = \{\langle x, 0, 1\rangle | x \in E\},$$

$$U^* = \{\langle x, 0, 0\rangle | x \in E\},$$

$$E^* = \{\langle x, 1, 0\rangle | x \in E\}.$$

Then,

$$\begin{aligned} \mathcal{P}(O^*) &= \{O^*\}, \\ \mathcal{P}(E^*) &= \{A | A \subseteq E^*\}, \end{aligned}$$

where A is a fixed IFS.

Definition of the Multi-Dimensional Modal Topological Structures

Here, following, but modifying and extending the first paper on MTS, we will define the concept of a Multi-Dimensional MTS (MDMTS), or, when the number of the dimensions is fixed, e.g., as $s \geq 1$ for some natural number s – as s-dimensional MTS (s-DMTS).

Let us have the fixed sets $X_1, \cdots X_s$, where the natural number $s \geq 1$.

Let

$$\langle \mathcal{P}(X_1), \mathcal{E}, \zeta, *, \eta \rangle$$

$$\cdots$$

$$\langle \mathcal{P}(X_s), \mathcal{E}, \zeta, *, \eta \rangle$$

where for each r $(1 \leq r \leq s)$, the object $\langle \mathcal{P}(X_r), \mathcal{E}, \zeta, *, \eta \rangle$ is a MTS.

Now, following the definition of a topological structure and the definition for a MTS, we construct the object

$$\langle \mathcal{P}(X_1 \times \cdots \times X_s), \mathcal{E}, \zeta_s, *, \eta_s \rangle$$

that will denote as an s-Dimensional(χ, η_s)-Modal(φ, ζ_s)-Topological Structure (s-D(χ, η_s)-M(φ, ζ_s)-TS) or (more general) multi-Dimensional(χ, η_s)-Modal(φ, ζ_s)-Topological Structure (mD(χ, η_s)-M(φ, ζ_s)-TS), where for $A, B \in \mathcal{P}(X_1 \times \cdots \times X_s)$:

- $\zeta_s : (X_1 \times \cdots \times X_s) \times (X_1 \times \cdots \times X_s) \to X_1 \times \cdots \times X_s$ is an associative operation, being a generator of the function ζ in the following sense: if

$$\zeta_2(a_1, a_2) = \zeta(a_1, a_2) = a_1 a_2 = \prod_{i=1}^{2} a_i,$$

then

$$\zeta_s(a_1, \cdots a_s) = \prod_{i=1}^{s} a_i;$$

- $\mathcal{E}$ is a topological operator. If it is from closure (cl) type, then it must satisfy the conditions

Ct1 $\mathcal{E}(A\zeta_s B) = \mathcal{E}(A)\zeta_s\mathcal{E}(B)$,

Ct2 $A \subseteq \mathcal{E}(A)$,

Ct3 $\mathcal{E}(\mathcal{E}(A)) = \mathcal{E}(A)$,

Ct4 $\mathcal{E}(O_1 \times \cdots \times O_s) = O_1 \times \cdots \times O_s$, where O_r is the minimal element of $\mathcal{P}(X_r)$ for $1 \leq r \leq s$, $O = O_1 \times \cdots \times O_s$ is the minimal element of the set $\mathcal{P}(X_1 \times \cdots \times X_s)$. If it is from interior (in) type, then it must satisfy the conditions

It1 $\mathcal{E}(A\zeta_s B) = \mathcal{E}(A)\zeta_s\mathcal{E}(B)$,

It2 $\mathcal{E}(A) \subseteq A$,

It3 $\mathcal{E}(\mathcal{E}(A)) = \mathcal{E}(A)$,

It4 $\mathcal{E}(X_1 \times \cdots \times X_s) = X_1 \times \cdots \times X_s$;

- $\eta_s : (X_1 \times \cdots \times X_s) \times (X_1 \times \cdots \times X_s) \to X_1 \times \cdots \times X_s$ is an associative operation being a generator of the function η in the above sense: if

$$\eta_2(a_1, a_2) = \eta(a_1, a_2) = \min(a_1, a_2) = \min_{1 \leq i \leq 2} a_i,$$

then

$$\eta_s(a_1, \cdots a_s) = \min_{1 \leq i \leq s} a_i;$$

- * is a modal operator from cl- or in-type.

  If it is from closure (cl) type, then it must satisfy the conditions

  Cm1 $\mathcal{E}(A\eta_s B) = \mathcal{E}(A)\eta_s\mathcal{E}(B)$,

  Cm2 $A \subseteq \mathcal{E}(A)$,

  Cm3 $\mathcal{E}(\mathcal{E}(A)) = \mathcal{E}(A)$,

  If it is from interior (in) type, then it must satisfy the conditions

  Im1 $\mathcal{E}(A\eta_s B) = \mathcal{E}(A)\eta_s\mathcal{E}(B)$,

  Im2 $\mathcal{E}(A) \subseteq A$,

  Im3 $\mathcal{E}(\mathcal{E}(A)) = \mathcal{E}(A)$;

- $\chi, \varphi \in \{cl, in\}$,
- both operators (topological and modal) must satisfy the condition

$$*\mathcal{E}(A) = \mathcal{E}(*A). \qquad (*)$$

We can see immediately that when $s = 1$, $\zeta_1 = \zeta, \eta_1 = \eta$, we obtain the definition of the standard MTS.

We must mention immediately, having in mind that the temporal scale T is a set, then each temporal MTS can be interpreted as a 2-DMTS.

If we require as additional conditions the functions ζ_s and η_s to be commutative and for every i, j $(1 \leq i < j \leq s)$ and for each $A_1 \times \cdots \times A_s \in \mathcal{P}(X_1 \times \cdots \times X_s)$:

$$\mathcal{E}(A_1 \times \cdots A_i \times \cdots \times A_j \times \cdots \times A_s) = \mathcal{E}(A_1 \times \cdots A_j \times \cdots \times A_i \times \cdots \times A_s), \quad (**)$$

then the MDMTS will be called a commutative MDMTS.

Intuitionistic Fussy Multi-Dimensional Modal Topological Structures

As a point of departure, further we give a list of the existing Cartesian products over two IFSs.

Let E_1 and E_2 be two universes and let

$$
\begin{aligned}
A &= \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E_1\}, \\
B &= \{\langle y, \mu_B(y), \nu_B(y)\rangle | y \in E_2\},
\end{aligned}
$$

be two IFSs over E_1 and over E_2, respectively. Then:

$$
\begin{aligned}
A \times_1 B = \;& \{\langle\langle x, y\rangle, \mu_A(x).\mu_B(y), \nu_A(x).\nu_B(y)\rangle | x \in E_1 \& y \in E_2\},\\
A \times_2 B = \;& \{\langle\langle x, y\rangle, \mu_A(x) + \mu_B(y) - \mu_A(x).\mu_B(y), \nu_A(x).\nu_B(y)\rangle\\
& | x \in E_1 \& y \in E_2\},\\
A \times_3 B = \;& \{\langle\langle x, y\rangle, \mu_A(x).\mu_B(y), \nu_A(x) + \nu_B(y) - \nu_A(x).\nu_B(y)\rangle\\
& | x \in E_1 \& y \in E_2\},\\
A \times_4 B = \;& \{\langle\langle x, y\rangle, \min(\mu_A(x), \mu_B(y)), \max(\nu_A(x), \nu_B(y))\rangle\\
& | x \in E_1 \& y \in E_2\},\\
A \times_5 B = \;& \{\langle\langle x, y\rangle, \max(\mu_A(x), \mu_B(y)), \min(\nu_A(x), \nu_B(y))\rangle\\
& | x \in E_1 \& y \in E_2\},\\
A \times_6 B = \;& \{\langle\langle x, y\rangle, \tfrac{\mu_A(x)+\mu_B(y)}{2}, \tfrac{\nu_A(x)+\nu_B(y)}{2})\rangle | x \in E_1 \& y \in E_2\}.
\end{aligned}
$$

Of course, these products can be generalize for the case of s universes $E_1, \cdots E_s$ and s IFSs $A_1, \cdots A_s$ so that the IFS A_i is an IFS over E_i for $1 \leq i \leq s$, as follows

$$A_1 \times_1 \cdots \times_1 A_s = \{\langle\langle x_1, \cdots x_s\rangle, \prod_{i=1}^{s} \mu_{A_i}(x_i), \prod_{i=1}^{s} \nu_{A_i}(x_i)\rangle | x_i \in E_i \text{ for } 1 \le i \le s\},$$

$$A_1 \times_2 \cdots \times_2 A_s = \{\langle\langle x_1, \cdots x_s\rangle, 1 - \prod_{i=1}^{s}(1 - \mu_{A_i}(x_i)), \prod_{i=1}^{s} \nu_{A_i}(x_i)\rangle | x_i \in E_i \text{ for } 1 \le i \le s\},$$

$$A_1 \times_3 \cdots \times_3 A_s = \{\langle\langle x_1, \cdots x_s\rangle, \prod_{i=1}^{s} \mu_{A_i}(x_i), 1 - \prod_{i=1}^{s}(1 - \nu_{A_i}(x_i))\rangle | x_i \in E_i \text{ for } 1 \le i \le s\},$$

$$A_1 \times_4 \cdots \times_4 A_s = \{\langle\langle x_1, \cdots x_s\rangle, \min_{1 \le i \le s} \mu_{A_i}(x_i), \max_{1 \le i \le s} \nu_{A_i}(x_i)\rangle | x_i \in E_i \text{ for } 1 \le i \le s\},$$

$$A_1 \times_5 \cdots \times_5 A_s = \{\langle\langle x_1, \cdots x_s\rangle, \max_{1 \le i \le s} \mu_{A_i}(x_i), \min_{1 \le i \le s} \nu_{A_i}(x_i)\rangle | x_i \in E_i \text{ for } 1 \le i \le s\},$$

$$A_1 \times_6 \cdots \times_6 A_s = \{\langle\langle x_1, \cdots x_s\rangle, \frac{\sum_{i=1}^{s} \mu_{A_i}(x_i)}{s}, \frac{\sum_{i=1}^{s} \nu_{A_i}(x_i)}{s}\rangle | x_i \in E_i \text{ for } 1 \le i \le s\}$$

For each one of these products we can see that their μ- and ν-functions are commutative and for the first five products – that their functions are associative.

In addition, we must re-defined the operations $\cup$ and $\cap$, relations $\subseteq$ and $=$, and operators $\Box$ and $\Diamond$ for IFSs over s universes $E_1, \cdots E_s$:

$$A_1 \times \cdots \times A_s \cup B_1 \times \cdots \times B_s$$

$$= \{\langle\langle x_1, \cdots, x_s\rangle, \max(\mu_A(x_1, \cdots, x_s), \mu_B(x_1, \cdots, x_s)),$$

$$\min(\nu_A(x_1, \cdots, x_s), \nu_B(x_1, \cdots, x_s))\rangle | \langle x_1, \cdots, x_s\rangle \in E_1^* \times \cdots \times E_s^*\},$$

$$A_1 \times \cdots \times A_s \cap B_1 \times \cdots \times B_s$$

$$= \{\langle\langle x_1, \cdots, x_s\rangle, \min(\mu_A(x_1, \cdots, x_s), \mu_B(x_1, \cdots, x_s)),$$

$$\max(\nu_A(x_1, \cdots, x_s), \nu_B(x_1, \cdots, x_s))\rangle | \langle x_1, \cdots, x_s\rangle \in E_1^* \times \cdots \times E_s^*\},$$

$A_1 \times \cdots \times A_s \subseteq B_1 \times \cdots \times B_s$ if and only if

$$= \forall \langle x_1, \cdots, x_s\rangle : \mu_A(x_1, \cdots, x_s) \leq \mu_B(x_1, \cdots, x_s)$$
$$\& \ \nu_A(x_1, \cdots, x_s) \geq \nu_B(x_1, \cdots, x_s),$$

$$A_1 \times \cdots \times A_s = B_1 \times \cdots \times B_s \text{ if and only if}$$

$$= \forall \langle x_1, \cdots, x_s \rangle : \mu_A(x_1, \cdots, x_s) = \mu_B(x_1, \cdots, x_s)$$
$$\& \, \nu_A(x_1, \cdots, x_s) = \nu_B(x_1, \cdots, x_s),$$

$$\square \, (A_1 \times \cdots \times A_s) = \{\langle \langle x_1, \cdots, x_s \rangle, \mu_A(x_1, \cdots, x_s), 1 - \mu_A(x_1, \cdots, x_s) \rangle |$$
$$\langle x_1, \cdots, x_s \rangle \in E_1^* \times \cdots \times E_s^* \},$$

$$\Diamond (A_1 \times \cdots \times A_s) = \{\langle \langle x_1, \cdots, x_s \rangle, 1 - \nu_A(x_1, \cdots, x_s), \nu_A(x_1, \cdots, x_s) \rangle |$$
$$\langle x_1, \cdots, x_s \rangle \in E_1^* \times \cdots \times E_s^* \}.$$

Now, we will construct some examples of MDMTSs using IFSs. They will be MDMTS-cases, or more exact, s-DMTS-cases of standard MTSs.

We must mention that when the topological structure from the present form is related to IFSs, it will be denoted as

IFs-D$(cl, \cap)$-M$(in, \cap)$-TS.

Let the μ- and ν-functions have one of the six forms discussed above.

Theorem 1. Let $E_1, \cdots, E_s$ be fixed universes. Then $\langle \mathcal{P}(E_1^* \times \cdots \times E_s^*), \mathcal{C}, \cup, \diamondsuit, \cap \rangle$ is an s-D$(cl, \cap)$-M$(cl, \cup)$-TS.

Proof.

First, we will give a detailed proof and after this will comment other forms of the proof.

Let $A_1 \times \cdots \times A_s, B_1 \times \cdots \times B_s \in \mathcal{P}(E_1^* \times \cdots \times E_s^*)$. Then, we check sequentially

Ct1:

$$\mathcal{C}(A_1 \times \cdots \times A_s \cup B_1 \times \cdots \times B_s)$$

$$= \mathcal{C}(\{\langle\langle x_1, \cdots, x_s\rangle, \mu_A(x_1, \cdots, x_s), \nu_A(x_1, \cdots, x_s)\rangle$$

$$|\langle x_1, \cdots, x_s\rangle \in E_1^* \times \cdots \times E_s^*\}$$

$$\cup \{\langle\langle x_1, \cdots, x_s\rangle, \mu_B(x_1, \cdots, x_s), \nu_B(x_1, \cdots, x_s)\rangle$$

$$|\langle x_1, \cdots, x_s\rangle \in E_1^* \times \cdots \times E_s^*\})$$

$$= \mathcal{C}(\{\langle\langle x_1, \cdots, x_s\rangle, \max(\mu_A(x_1, \cdots, x_s), \mu_B(x_1, \cdots, x_s)),$$

$$\min(\nu_A(x_1, \cdots, x_s), \nu_B(x_1, \cdots, x_s))\rangle | \langle x_1, \cdots, x_s\rangle \in E_1^* \times \cdots \times E_s^*\})$$

$$
\begin{aligned}
&= \{\langle\langle x_1,\cdots,x_s\rangle, \sup_{\langle y_1,\cdots,y_s\rangle\in E_1^*\times\cdots\times E_s^*} \max(\mu_A(y_1,\cdots,y_s),\mu_B(y_1,\cdots,y_s)),\\
&\qquad \inf_{\langle y_1,\cdots,y_s\rangle\in E_1^*\times\cdots\times E_s^*} \min(\nu_A(y_1,\cdots,y_s),\nu_B(y_1,\cdots,y_s))\rangle\\
&\qquad |\langle x_1,\cdots,x_s\rangle \in E_1^*\times\cdots\times E_s^*\}\\
&= \{\langle\langle x_1,\cdots,x_s\rangle, \max(\sup_{\langle y_1,\cdots,y_s\rangle\in E_1^*\times\cdots\times E_s^*} \mu_A(y_1,\cdots,y_s), \sup_{\langle y_1,\cdots,y_s\rangle\in E_1^*\times\cdots\times E_s^*} \mu\\
&\qquad \min(\inf_{\langle y_1,\cdots,y_s\rangle\in E_1^*\times\cdots\times E_s^*} \min(\nu_A(y_1,\cdots,y_s), \inf_{\langle y_1,\cdots,y_s\rangle\in E_1^*\times\cdots\times E_s^*} \nu_B(y_1,\cdot\cdot\\
&\qquad |\langle x_1,\cdots,x_s\rangle \in E_1^*\times\cdots\times E_s^*\}\\
&= \mathcal{C}(A)\cup\mathcal{C}(B);
\end{aligned}
$$

Ct2:

$$A_1 \times \cdots \times A_s$$

$$= \{\langle\langle x_1, \cdots, x_s\rangle, \mu_A(x_1, \cdots, x_s), \nu_A(x_1, \cdots, x_s)\rangle |$$

$$\langle x_1, \cdots, x_s\rangle \in E_1^* \times \cdots \times E_s^*\}$$

$$\subseteq \{\langle\langle x_1, \cdots, x_s\rangle, \sup_{\langle y_1, \cdots, y_s\rangle \in E_1^* \times \cdots \times E_s^*} \mu_A(y_1, \cdots, y_s),$$

$$\inf_{\langle y_1, \cdots, y_s\rangle \in E_1^* \times \cdots \times E_s^*} \min(\nu_A(y_1, \cdots, y_s))\rangle | \langle x_1, \cdots, x_s\rangle \in E_1^* \times \cdots \times E_s^*\}$$

$$= \mathcal{C}(A_1 \times \cdots \times A_s);$$

Ct3: Having in mind that $\sup\limits_{\langle y_1,\cdots,y_s\rangle\in E_1^*\times\cdots\times E_s^*} \mu_A(y_1,\cdots,y_s)$ and $\inf\limits_{\langle y_1,\cdots,y_s\rangle\in E_1^*\times\cdots\times E_s^*} \nu_A(y_1,\cdots,y_s)$ are constants, we obtain:

$$
\begin{aligned}
&\mathcal{C}(\mathcal{C}(A_1\times\cdots\times A_s))\\
&=\mathcal{C}(\{\langle\langle x_1,\cdots,x_s\rangle, \sup_{\langle y_1,\cdots,y_s\rangle\in E_1^*\times\cdots\times E_s^*} \mu_A(y_1,\cdots,y_s),\\
&\qquad \inf_{\langle y_1,\cdots,y_s\rangle\in E_1^*\times\cdots\times E_s^*} \nu_A(y_1,\cdots,y_s))\rangle | \langle x_1,\cdots,x_s\rangle \in E_1^*\times\cdots\times E_s^*\})\\
&=\{\langle\langle x_1,\cdots,x_s\rangle, \sup_{\langle y_1,\cdots,y_s\rangle\in E_1^*\times\cdots\times E_s^*} \mu_A(y_1,\cdots,y_s),\\
&\qquad \inf_{\langle y_1,\cdots,y_s\rangle\in E_1^*\times\cdots\times E_s^*} \nu_A(y_1,\cdots,y_s))\rangle | \langle x_1,\cdots,x_s\rangle \in E_1^*\times\cdots\times E_s^*\}\\
&=\mathcal{C}(A_1\times\cdots\times A_s);
\end{aligned}
$$

Ct4:

$$\mathcal{C}(O_1^* \times \cdots \times O_s^*)$$

$$= \mathcal{C}(\{\langle\langle x_1, \cdots, x_s\rangle, 0, 1\rangle | \langle x_1, \cdots, x_s\rangle \in E_1^* \times \cdots \times E_s^*\}\})$$

$$= \{\langle\langle x_1, \cdots, x_s\rangle, \sup_{\langle y_1, \cdots, y_s\rangle \in E_1^* \times \cdots \times E_s^*} 0,$$

$$\inf_{\langle y_1, \cdots, y_s\rangle \in E_1^* \times \cdots \times E_s^*} 1\rangle | \langle x_1, \cdots, x_s\rangle \in E_1^* \times \cdots \times E_s^*\}$$

$$\{\langle\langle x_1, \cdots, x_s\rangle, 0, 1\rangle | \langle x_1, \cdots, x_s\rangle \in E_1^* \times \cdots \times E_s^*\}\})$$

$$= O_1^* \times \cdots \times O_s^*;$$

Cm1:

$$\Diamond(A_1 \times \cdots \times A_s \cap B_1 \times \cdots \times B_s) = \Diamond(\{\langle\langle x_1, \cdots, x_s\rangle, \mu_A(x_1, \cdots, x_s),$$
$$\nu_A(x_1, \cdots, x_s)\rangle | \langle x_1, \cdots, x_s\rangle \in E_1^* \times \cdots \times E_s^*\}$$
$$\cap \{\langle\langle x_1, \cdots, x_s\rangle, \mu_B(x_1, \cdots, x_s), \nu_B(x_1, \cdots, x_s)\rangle$$
$$|\langle x_1, \cdots, x_s\rangle \in E_1^* \times \cdots \times E_s^*\})$$
$$= \Diamond\{\langle\langle x_1, \cdots, x_s\rangle, \min(\mu_A(x_1, \cdots, x_s), \mu_B(x_1, \cdots, x_s)),$$
$$\max(\nu_A(x_1, \cdots, x_s), \nu_B(x_1, \cdots, x_s))\rangle | \langle x_1, \cdots, x_s\rangle \in E_1^* \times \cdots \times E_s^*\}$$
$$= \{\langle\langle x_1, \cdots, x_s\rangle, 1 - \max(\nu_A(x_1, \cdots, x_s), \nu_B(x_1, \cdots, x_s)),$$
$$\max(\nu_A(x_1, \cdots, x_s), \nu_B(x_1, \cdots, x_s))\rangle | \langle x_1, \cdots, x_s\rangle \in E_1^* \times \cdots \times E_s^*\}$$
$$= \{\langle\langle x_1, \cdots, x_s\rangle, \min(1 - \nu_A(x_1, \cdots, x_s), 1 - \nu_B(x_1, \cdots, x_s)),$$
$$\max(\nu_A(x_1, \cdots, x_s), \nu_B(x_1, \cdots, x_s))\rangle | \langle x_1, \cdots, x_s\rangle \in E_1^* \times \cdots \times E_s^*\}$$
$$= \{\langle\langle x_1, \cdots, x_s\rangle, 1 - \nu_A(x_1, \cdots, x_s), \nu_A(x_1, \cdots, x_s)\rangle$$
$$|\langle x_1, \cdots, x_s\rangle \in E_1^* \times \cdots \times E_s^*\}$$
$$\cap \{\langle\langle x_1, \cdots, x_s\rangle, 1 - \nu_B(x_1, \cdots, x_s), \nu_B(x_1, \cdots, x_s)\rangle$$
$$|\langle x_1, \cdots, x_s\rangle \in E_1^* \times \cdots \times E_s^*\}$$
$$= \Diamond(A_1 \times \cdots \times A_s) \cap \Diamond(B_1 \times \cdots \times B_s);$$

Cm2:

$$
\begin{aligned}
& A_1 \times \cdots \times A_s \\
& = \{\langle\langle x_1, \cdots, x_s\rangle, \mu_A(x_1, \cdots, x_s), \nu_A(x_1, \cdots, x_s)\rangle \\
& \quad |\langle x_1, \cdots, x_s\rangle \in E_1^* \times \cdots \times E_s^*\} \\
& \subseteq \{\langle\langle x_1, \cdots, x_s\rangle, 1 - \nu_A(x_1, \cdots, x_s), \nu_A(x_1, \cdots, x_s)\rangle \\
& \quad |\langle x_1, \cdots, x_s\rangle \in E_1^* \times \cdots \times E_s^*\} \\
& = \Diamond(A_1 \times \cdots \times A_s);
\end{aligned}
$$

Cm3:

$$
\begin{aligned}
&\Diamond(\Diamond(A_1 \times \cdots \times A_s)) \\
&= \Diamond(\{\langle\langle x_1, \cdots, x_s\rangle, 1 - \nu_A(x_1, \cdots, x_s), \nu_A(x_1, \cdots, x_s)\rangle \\
&\quad |\langle x_1, \cdots, x_s\rangle \in E_1^* \times \cdots \times E_s^*\}) \\
&= \{\langle\langle x_1, \cdots, x_s\rangle, 1 - \nu_A(x_1, \cdots, x_s), \nu_A(x_1, \cdots, x_s)\rangle \\
&\quad |\langle x_1, \cdots, x_s\rangle \in E_1^* \times \cdots \times E_s^*\} \\
&= \Diamond(A_1 \times \cdots \times A_s);
\end{aligned}
$$

(*):

$$
\begin{aligned}
&\Diamond(\mathcal{C}(A_1 \times \cdots \times A_s)) \\
&= \Diamond(\{\langle\langle x_1, \cdots, x_s\rangle, \sup_{\langle y_1,\cdots,y_s\rangle \in E_1^* \times \cdots \times E_s^*} \mu_A(y_1, \cdots, y_s), \\
&\qquad \inf_{\langle y_1,\cdots,y_s\rangle \in E_1^* \times \cdots \times E_s^*} \nu_A(y_1, \cdots, y_s)\rangle | \langle x_1, \cdots, x_s\rangle \in E_1^* \times \cdots \times E_s^*\}) \\
&= \{\langle\langle x_1, \cdots, x_s\rangle, 1 - \inf_{\langle y_1,\cdots,y_s\rangle \in E_1^* \times \cdots \times E_s^*} \nu_A(y_1, \cdots, y_s), \\
&\qquad \inf_{\langle y_1,\cdots,y_s\rangle \in E_1^* \times \cdots \times E_s^*} \nu_A(y_1, \cdots, y_s)\rangle | \langle x_1, \cdots, x_s\rangle \in E_1^* \times \cdots \times E_s^*\} \\
&= \{\langle\langle x_1, \cdots, x_s\rangle, \sup_{\langle y_1,\cdots,y_s\rangle \in E_1^* \times \cdots \times E_s^*} 1 - \nu_A(y_1, \cdots, y_s), \\
&\qquad \inf_{\langle y_1,\cdots,y_s\rangle \in E_1^* \times \cdots \times E_s^*} \nu_A(y_1, \cdots, y_s)\rangle | \langle x_1, \cdots, x_s\rangle \in E_1^* \times \cdots \times E_s^*\} \\
&= \mathcal{C}(\Diamond(A_1 \times \cdots \times A_s));
\end{aligned}
$$

(**): This condition is valid because as we mentioned above, the μ- and ν-functions have one of the seven forms and all of them are commutative.

This completes the proof. □

In the same manner we can also prove the following theorems.

Theorem 2. Let $E_1, \cdots, E_s$ be fixed universes. Then $\langle \mathcal{P}(E_1^* \times \cdots \times E_s^*), \mathcal{C}, \cup, \diamondsuit, \cup \rangle$ is an s-D$(cl, \cup)$-M$(cl, \cup)$-TS.

Theorem 3. Let $E_1, \cdots, E_s$ be fixed universes. Then $\langle \mathcal{P}(E_1^* \times \cdots \times E_s^*), \mathcal{C}, \cup, \square, \cap \rangle$ is an s-D$(in, \cap)$-M$(cl, \cup)$-TS.

Theorem 4. Let $E_1, \cdots, E_s$ be fixed universes. Then $\langle \mathcal{P}(E_1^* \times \cdots \times E_s^*), \mathcal{C}, \cup, \square, \cup \rangle$ is an s-D$(in, \cup)$-M$(cl, \cup)$-TS.

Theorem 5. Let $E_1, \cdots, E_s$ be fixed universes. Then $\langle \mathcal{P}(E_1^* \times \cdots \times E_s^*), \mathcal{I}, \cup, \diamondsuit, \cap \rangle$ is an s-D$(cl, \cap)$-M$(in, \cup)$-TS.

Theorem 6. Let $E_1, \cdots, E_s$ be fixed universes. Then $\langle \mathcal{P}(E_1^* \times \cdots \times E_s^*), \mathcal{I}, \cup, \diamondsuit, \cup \rangle$ is an s-D$(cl, \cup)$-M$(in, \cup)$-TS.

Theorem 7. Let $E_1, \cdots, E_s$ be fixed universes. Then $\langle \mathcal{P}(E_1^* \times \cdots \times E_s^*), \mathcal{I}, \cup, \square, \cap \rangle$ is an s-D$(in, \cap)$-M$(in, \cup)$-TS.

Theorem 8. Let $E_1, \cdots, E_s$ be fixed universes. Then $\langle \mathcal{P}(E_1^* \times \cdots \times E_s^*), \mathcal{I}, \cup, \square, \cup \rangle$ is an s-D$(in, \cup)$-M$(in, \cup)$-TS.

Finally, we will mention that

$$\mathcal{C}(U_1 \times \cdots \times U_s) = U_1 \times \cdots \times U_s,$$

and

$$\mathcal{I}(U_1 \times \cdots \times U_s) = U_1 \times \cdots \times U_s.$$

Conclusion

By analogy with previous research, let us imagine that each of the structures $\langle \mathcal{P}(X_1), \mathcal{E}, \zeta, *, \eta \rangle, \cdots, \langle \mathcal{P}(X_s), \mathcal{E}, \zeta, *, \eta \rangle$ is a page of a book, where each page is enumerated with the number of the structure located on it. Then the common components of any of the structures can be interpreted as the spine of the book, and the entire mDMTS – as the book itself, as it is shown on the Fig. 1. Now, each page of our book can be interpreted as a map and then the whole book can be interpreted as an atlas.

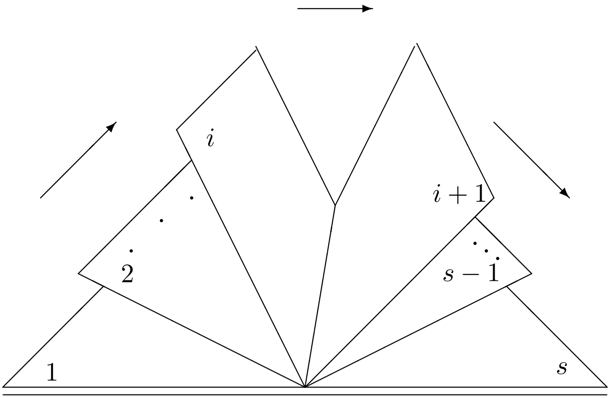

Fig. 1. Atlas with maps generated by the structures $\langle \mathcal{P}(X_1), \mathcal{E}, \zeta, *, \eta \rangle, \cdots, \langle \mathcal{P}(X_s), \mathcal{E}, \zeta, *, \eta \rangle$

We will finish with the following three **Open problems**:

1. Which other topological operators are suitable for generating of mDMTSs?
2. Which other logical operators are suitable for generating of mDMTSs?
3. Which other types of mDMTSs can be constructed?

Acknowledgement

The research is supported by the Bulgarian National Science Fund under Grant Ref. No. KP-06-72/8 “Intuitionistic fuzzy methods for data analysis with an emphasis on the blood donation system in Bulgaria".

Conclusion

In a next research, we will give the list of the intuitionistic fuzzy conjunctions and disjunctions generated by the above implications and negations and will discuss some of their properties.

Acknowledgement
This research was funded by the Bulgarian National Science Fund, grant number KP-06-N22/1/2018 "Theoretical research and applications of InterCriteria Analysis".

Thank you for attention!