A general approach to feeble modal topological structures illustrated by intuitionistic fuzzy objects

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Introduction

During the last two years, in a series of papers, the author introduced the concept of a Modal Topological Structure (MTS). Its basic properties and some of its modifications have been studied, and it has been illustrated through intuitionistic fuzzy objects — intuitionistic fuzzy sets (IFSs) and operations, relations and operators, defined over them.

Modal Topological Structures

Here, we will define the concept of a MTS.
Let us have a fixed set X and let everywhere below

$$\mathcal{P}(X) = \{Y | Y \subseteq X\}.$$

Let O be the minimal element of $\mathcal{P}(X)$ and let for $Y \in \mathcal{P}(X)$,

$$\neg Y = X - Y,$$

where " $-$ " is the set-theoretical operation "subtraction".
Let us have two operations $\Delta, \nabla : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathcal{P}(X)$ defined in such a way that for every two sets $A, B \in \mathcal{P}(X)$:

$$A \nabla B = \neg(\neg A \Delta \neg B), \quad (1)$$

$$A \Delta B = \neg(\neg A \nabla \neg B). \quad (2)$$

Let the operation Δ generate the topological operator $\mathcal{O}$ and the operation ∇ generate the topological operator $\mathcal{Q}$.
Now, following Kuratowski's definiton we will say that the operator $\mathcal{O}$ is a closure (cl)-topological operator if for each $A, B \in \mathcal{P}(X)$:

C1 $\mathcal{O}(A \Delta B) = \mathcal{O}(A) \Delta \mathcal{O}(B)$,

C2 $A \subseteq \mathcal{O}(A)$,

C3 $\mathcal{O}(\mathcal{O}(A)) = \mathcal{O}(A)$,

C4 $\mathcal{O}(O) = O$,

and that the operator $\mathcal{Q}$ is an interior (in)-topological operator if for each $A, B \in \mathcal{P}(X)$:

I1 $\mathcal{Q}(A \nabla B) = \mathcal{Q}(A) \nabla \mathcal{Q}(B)$,

I2 $\mathcal{Q}(A) \subseteq A$,

I3 $\mathcal{Q}(\mathcal{Q}(A)) = \mathcal{Q}(A)$,

I4 $\mathcal{Q}(X) = X$.

We will assume that for every set $A \in \mathcal{P}(X)$:

$$\mathcal{O}(A) = \neg(\mathcal{Q}(\neg A)), \quad (3)$$

$$\mathcal{Q}(A) = \neg(\mathcal{O}(\neg A)). \quad (4)$$

Having in mind that the modal operators $\Diamond$ and $\Box$ satisfy conditions C1 - C3 and I1 - I3, respectively, we see that operator $\Diamond$ is from a cl-type and operation $\Box$ is from an in-type. More general, if the modal operator $\circ$ is from cl-type, i.e., if it satisfies conditions C1 - C3 with symbols $\circ$ instead of $\mathcal{O}$, and if the modal operator $\bullet$ is from in-type, i.e., if it satisfies conditions I1 - I3 with symbol $\bullet$ instead of $\mathcal{Q}$, then we will assume that for each $A \in \mathcal{P}(X)$

$$\circ A = \neg(\bullet \neg A), \tag{7}$$

$$\bullet A = \neg(\circ \neg A). \tag{8}$$

Because the topological structures can be from cl- or in-type, we can mention that a given structure is from α-type, where $\alpha \in \{cl, in\}$.
Now, following Bourbaki's definition of a topological structure, we define that the object $\langle \mathcal{P}(X), \mathcal{E}, \zeta, *, \eta \rangle$ is a β-Modal α-Topological Structure (β-Mα-TS) over the set X, where $\mathcal{E} \in \{\mathcal{O}, \mathcal{Q}\}$ is a topological operator from α-type generated by operation $\zeta \in \{\Delta, \nabla\}$ and $* \in \{\circ, \bullet\}$ is a modal operator from β-type related with operation $\eta \in \{\Delta, \nabla\}$, where $\alpha, \beta \in \{cl, in\}$. Therefore, each one of the operators (the topological and the modal) must satisfy the respective C- or the respective I-conditions.
In addition, the two types of operators must satisfy the following additional condition (*) for each $A \in \mathcal{P}(X)$:

$$*\mathcal{E}(A) = \mathcal{E}(*A). \qquad (*)$$

On the Feeble Modal Topological Structures

Obviously, relation $A = B$ is a particular case of relations $A \subseteq B$ and $A \supseteq B$ for every two objects A and B that are comparable with these relations. But when there is a condition according to which the relation is $=$, its substitution with one of the two relations $\subseteq$ and $\supseteq$ leads to a weak condition. This situation appeared after the publishing of the first paper over MTS. In some of the previous papers, we constructed structures that do not satisfy conditions C1 - C4 or I1 - I4 in the form of Chapter 3: in some of the cases, instead of relation “$=$" one of the relations for inclusion was valid. In other cases, there was no respective relation. Therefore, in each one of these cases, we obtained structures with weak properties. On the other hand, in topology the word “weak" has different sense than the one discussed here.

By this reason, in our papers, we started using the word “feeble" instead of “weak" for the respective MTSs. In the next sections, we will describe some Feeble MTS (FMTS) and will illustrate them with examples from the area of intuitionistic fuzziness, constructing Intuitionistic Fuzzy FMTSs (IFFMTSs).
Let for $0 \leq s \leq 4, 0 \leq t \leq 3$:
$J_1, \ldots J_s \in \{C1, C2, C3, C4, I1, I2, I3, I4\}$, $K_1, \ldots, K_t \in \{C1, C2, C3, I1, I2, I3\}, R, S_1, \ldots, S_s, T_1, \ldots, T_t, \in \{\subseteq, \supseteq, \perp\}$,
where “$\perp$" denotes the empty symbol that will be used to denote that the respective condition is not valid.

Now, we will define that the object $\langle \mathcal{P}(X), \mathcal{E}, \zeta, *, \eta \rangle$ is a R-Feeble$(\beta, \eta; K_1T_1, \ldots K_tT_t)$-Modal $(\alpha; J_1S_1, \ldots, J_sS_s)$-Topological Structure (with abbreviation: R-F$(\beta, \eta; K_1T_1, \ldots K_tT_t)$-M$(\alpha; J_1S_1, \ldots, J_sS_s)$-TS) over the set X, where $\mathcal{E} \in \{\mathcal{O}, \mathcal{Q}\}$ is a topological operator from α-type generated by operation $\zeta \in \{\Delta, \nabla\}$ and $* \in \{\circ, \bullet\}$ is a modal operator from β-type related with operation $\eta \in \{\Delta, \nabla\}$, where $\alpha, \beta \in \{cl, in\}$, so that each one of the both operators (the topological and the modal) must satisfy the respective C- or the respective I-conditions and in condition J_i the original relation from Section 2 is replaced with the relation S_i, or this condition is omitted $(1 \leq i \leq s)$ and in condition K_j the original relation is replaced with the relation T_j or this condition is omitted $(1 \leq j \leq t)$. Similarly, by $R \in \{\subseteq, \supseteq\}$ we denote the fact that in equality (*) symbol $=$ is replaced with relation R. We assume that if R is $\perp$, then the relation is the original one and R will be omitted.

When s or t is equal to 0, then in the FMTS there is no change in the first or second group of conditions and the respective group of symbols is omitted.

IFMTSs with intuitionistic fuzzy modal operators $\mathcal{C}_{33}$ and $\mathcal{I}_{33}$

First, we give the definitions of the following two intuitionistic fuzzy operations.

$$
\begin{aligned}
A \cap_{33} B &= \{\langle x, \min(\mu_A(x), \mu_B(x)), 1 - \min(\mu_A(x), \mu_B(x))\rangle | x \in E\}; \\
A \cup_{33} B &= \{\langle x, 1 - \min(\nu_A(x), \nu_B(x)), \min(\nu_A(x), \nu_B(x))\rangle | x \in E\}.
\end{aligned}
$$

By analogy with the construction of the first two intuitionistic fuzzy topological operators $\mathcal{C}$ and $\mathcal{I}$ on the basis of the standard operations $\cup$ and $\cap$, in a previous paper we constructed two intuitionistic fuzzy topological operators:

$$\begin{aligned}\mathcal{C}_{33}(A) &= \{\langle x, 1 - \inf_{y \in E} \nu_A(y), \inf_{y \in E} \nu_A(y)\rangle | x \in E\}; \\ \mathcal{I}_{33}(A) &= \{\langle x, \inf_{y \in E} \mu_A(y), 1 - \inf_{y \in E} \mu_A(y)\rangle | x \in E\},\end{aligned}$$

for each IFS A.

In a previous paper, it is proved that $\langle \mathcal{P}(E^*), \mathcal{C}_{33}, \cup_{33}, \diamondsuit, \cup_{33}\rangle$ is an IFF$(cl, \cup_{33}, 1 \subseteq)$-M$cl$-TS and $\langle \mathcal{P}(E^*), \mathcal{I}_{33}, \cap_{33}, \Box, \cap_{33}\rangle$ is an IFF$(in, \cap_{33}, 1 \supseteq)$-M$in$-TS. Here, we discuss the remaining MTS that have similar form and for them we will prove that they are feeble ones.

Theorem 1. For each universe E, $\langle \mathcal{P}(E^*), \mathcal{C}_{33}, \cup_{33}, \diamondsuit, \cap_{33}\rangle$ is an IFF$(cl, \cap_{33}, 1 \subseteq)$-M$cl$-TS.

Proof. Let the IFSs $A, B \in \mathcal{P}(E^*)$ be given. Despite that a part of the checks of the conditions are given in a previous paper], we will give them here for completeness and the proofs of the next assertions will be omitted.

C1.

$$\mathcal{C}_{33}(A \cup_{33} B)$$
$$= \mathcal{C}_{33}(\{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\} \cup_{33} \{\langle x, \mu_B(x), \nu_B(x)\rangle | x \in E\})$$
$$= \mathcal{C}_{33}(\{\langle x, 1 - \min(\nu_A(x), \nu_B(x)), \min(\nu_A(x), \nu_B(x))\rangle | x \in E\})$$
$$= \{\langle x, 1 - \inf_{y \in E}(\min(\nu_A(x), \nu_B(x))), \inf_{y \in E} \min(\nu_A(x), \nu_B(x))\rangle | x \in E\}$$
$$= \{\langle x, 1 - \min(\inf_{y \in E} \nu_A(x), \inf_{y \in E} \nu_B(x)), \min(\inf_{y \in E} \nu_A(x), \inf_{y \in E} \nu_B(x))\rangle | x \in E\}$$
$$= \{\langle x, 1 - \inf_{y \in E} \nu_A(x), \inf_{y \in E} \nu_A(x)\rangle | x \in E\}$$
$$\cup_{33} \{\langle x, 1 - \inf_{y \in E} \nu_B(x), \inf_{y \in E} \nu_B(x)\rangle | x \in E\}$$
$$= \mathcal{C}_{33}(A) \cup_{33} \mathcal{C}_{33}(B);$$

C2.

$$\begin{aligned} A &= \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\} \\ &\subseteq \{\langle x, \sup_{y\in E} \mu_A(x), \inf_{y\in E} \nu_A(x)\rangle | x \in E\} \\ &\subseteq \{\langle x, \sup_{y\in E}(1 - \nu_A(x)), \inf_{y\in E} \nu_A(x)\rangle | x \in E\} \\ &= \{\langle x, 1 - \inf_{y\in E} \nu_A(x), \inf_{y\in E} \nu_A(x)\rangle | x \in E\} \\ &= \mathcal{C}_{33}(A); \end{aligned}$$

C3. Having in mind that $\inf_{y\in E} \nu_A(y)$ is a constant, we obtain:

$$\begin{aligned}
\mathcal{C}_{33}(\mathcal{C}_{33}(A)) &= \mathcal{C}_{33}(\{\langle x, 1 - \inf_{y\in E} \nu_A(x), \inf_{y\in E} \nu_A(x)\rangle | x \in E\}) \\
&= \{\langle x, 1 - \inf_{z\in E} \inf_{y\in E} \nu_A(y), \inf_{z\in E} \inf_{y\in E} \nu_A(y)\rangle | x \in E\} \\
&= \{\langle x, 1 - \inf_{y\in E} \nu_A(y), \inf_{y\in E} \nu_A(y)\rangle | x \in E\} \\
&= \mathcal{C}_{33}(A);
\end{aligned}$$

C4.

$$\begin{aligned}
\mathcal{C}_{33}(O^*) &= \mathcal{C}_{33}(\{\langle x, 0, 1\rangle | x \in E\}) \\
&= \{\langle x, 1 - \inf_{y\in E} 1, \inf_{y\in E} 1\rangle | x \in E\} \\
&= \{\langle x, 0, 1\rangle | x \in E\} = O^*;
\end{aligned}$$

C1.

$$
\begin{aligned}
&\Diamond(A \cap_{33} B) \\
&= \Diamond(\{\langle x, \min(\mu_A(x), \mu_B(x)), 1 - \min(\mu_A(x), \mu_B(x))\rangle | x \in E\}); \\
&= \{\langle x, \min(\mu_A(x), \mu_B(x)), 1 - \min(\mu_A(x), \mu_B(x))\rangle | x \in E\}; \\
&\subseteq \{\langle x, \min(1 - \nu_A(x), 1 - \nu_B(x)), 1 - \min(\mu_A(x), \mu_B(x))\rangle | x \in E\}; \\
&\subseteq \{\langle x, \min(1 - \nu_A(x), 1 - \nu_B(x)), \\
&\qquad 1 - \min(1 - \nu_A(x), 1 - \nu_B(x))\rangle | x \in E\}; \\
&= \{\langle x, 1 - \nu_A(x), \nu_A(x)\rangle | x \in E\} \cap_{33} \{\langle x, 1 - \nu_B(x), \nu_B(x)\rangle | x \in E\} \\
&= \Diamond A \cap_{33} \Diamond B.
\end{aligned}
$$

C2:

$$
\begin{aligned}
A &= \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\} \\
&\subseteq \{\langle x, 1-\nu_A(x), \nu_A(x)\rangle | x \in E\} \\
&= \Diamond A;
\end{aligned}
$$

C3:

$$
\begin{aligned}
\Diamond\Diamond A &= \Diamond\{\langle x, 1-\nu_A(x), \nu_A(x)\rangle | x \in E\} \\
&= \{\langle x, 1-\nu_A(x), \nu_A(x)\rangle | x \in E\} \\
&= \Diamond A;
\end{aligned}
$$

C4:

$$
\begin{aligned}
\Diamond O^* &= \Diamond\{\langle x, 0, 1\rangle | x \in E\} \\
&= O^*;
\end{aligned}
$$

(*)

$$\begin{aligned}\Diamond\mathcal{C}_{33}(A) &= \Diamond\{\langle x, 1-\inf_{y\in E}\nu_A(x), \inf_{y\in E}\nu_A(x)\rangle | x\in E\}\\ &= \{\langle x, 1-\inf_{y\in E}\nu_A(x), \inf_{y\in E}\nu_A(x)\rangle | x\in E\}\\ &= \mathcal{C}_{33}(\{\langle x, 1-\nu_A(x), \nu_A(x)\rangle | x\in E\})\\ &= \mathcal{C}_{33}(\Diamond A).\end{aligned}$$

This completes the proof. □

By the same way we can prove

Theorem 2. For each universe E, $\langle \mathcal{P}(E^*), \mathcal{C}_{33}, \cup_{33}, \Box, \cup_{33} \rangle$ is an IFF$(in, \cup_{33}, 1 \supseteq)$-M$cl$-TS.

Theorem 3. For each universe E, $\langle \mathcal{P}(E^*), \mathcal{I}_{33}, \cap_{33}, \Diamond, \cap_{33} \rangle$ is an IFF$(cl, \cap_{33}, 1 \subseteq)$-M$cl$-TS.

Theorem 4. For each universe E, $\langle \mathcal{P}(E^*), \mathcal{I}_{33}, \cap_{33}, \Box, \cup_{33} \rangle$ is an IFF$(in, \cup_{33}, 1 \supseteq)$-M$cl$-TS.

Bellow, we omit the proofs of the next theorem, because they are similar to the above one.

Intuitionistic Fuzzy Feeble Modal Topological Structures with operators $\mathcal{C}_{12}$ and $\mathcal{I}_{12}$

In this Section, we will illustrate the FMTSs with four examples using the following intuitionistic fuzzy operations and topological operators:

$$A \cup_{12} B = \{\langle x, \max(\mu_A(x), \mu_B(x)), 1 - \max(\mu_A(x), \mu_B(x))\rangle | x \in E\},$$
$$A \cap_{12} B = \{\langle x, 1 - \max(\nu_A(x), \nu_B(x)), \max(\nu_A(x), \nu_B(x))\rangle | x \in E\}.$$

$$\mathcal{C}_{12}(A) = \{\langle x, \sup_{y \in E} \mu_A(y), 1 - \sup_{y \in E} \mu_A(y)\rangle | x \in E\};$$
$$\mathcal{I}_{12}(A) = \{\langle x, 1 - \sup_{y \in E} \nu_A(y), \sup_{y \in E} \nu_A(y)\rangle | x \in E\};$$

Theorem 5. For each universe E, $\langle \mathcal{P}(E^*), \mathcal{C}_{12}, \cup_{12}, \diamondsuit, \cap_{12} \rangle$ is an IF$\subseteq$-F$(cl, \cap_{12})$-Mcl-TS.

Theorem 6. For each universe E, $\langle \mathcal{P}(E^*), \mathcal{C}_{12}, \cup_{12}, \square, \cap_{12} \rangle$ is an IFF$(in, \cap_{12}; 1 \supseteq)$-M$cl$-TS.

Theorem 7. For each universe E, $\langle \mathcal{P}(E^*), \mathcal{I}_{12}, \cap_{12}, \diamondsuit, \cup_{12} \rangle$ is an IFF$(cl, \cup_{12}; 1 \subseteq)$-M$in$-TS.

Theorem 8. For each universe E, $\langle \mathcal{P}(E^*), \mathcal{I}_{12}, \cap_{12}, \square, \cup_{12} \rangle$ is an IF$\supseteq$-F$(in, \cup_{12}$-Min-TS.

Theorem 9. For each universe E, $\langle \mathcal{P}(E^*), \mathcal{C}_{12}, \cap_{12}, \diamondsuit, \cup_{12} \rangle$ is an IF$\subseteq$-F$(cl, \cup_{12}, 1 \subseteq)$-M$cl$-TS.

Theorem 10. For each universe E, $\langle \mathcal{P}(E^*), \mathcal{I}_{12}, \cap_{12}, \square, \cap_{12} \rangle$ is an IF$\supseteq$-F$(in, \cap_{12}, 1 \supseteq)$-M$in$-TS.

Intuitionistic Fuzzy Feeble Modal Topological Structures with operators $\mathcal{C}^*$ and $\mathcal{I}^*$

Below, we will discuss four IFFMTSs based of the operations $\cap^*$ and $\cup^*$, and operators $\mathcal{C}^*$ and $\mathcal{I}^*$ introduced for each IFS A, as follows:

$$A \cap^* B = \{\langle x, \mu_A(x)\mu_B(x), \max(\nu_A(x), \nu_B(x))\rangle | x \in E\},$$

$$A \cup^* B = \{\langle x, \max(\mu_A(x), \mu_B(x)), \nu_A(x)\nu_B(x)\rangle | x \in E\},$$

$$\mathcal{C}^*(A) = \{\langle x, \sup_{y \in E} \mu_A(y), \prod_{y \in E} \nu_A(y)\rangle | x \in E\},$$

$$\mathcal{I}^*(A) = \{\langle x, \prod_{y \in E} \mu_A(y), \sup_{y \in E} \nu_A(y)\rangle | x \in E\}.$$

Theorem 11. For each universe E, $\langle \mathcal{P}(E^*), \mathcal{C}^*, \cup^*, \diamondsuit, \cup^* \rangle$ is an IF$\supseteq$-F$(cl, \cup^*; 1 \supseteq)$-M$(cl; 3 \supseteq)$-TS.

Theorem 12. For each universe E, $\langle \mathcal{P}(E^*), \mathcal{I}^*, \cap^*, \square, \cap^* \rangle$ is an IF$\subseteq$-F$(in, \cap^*; 1 \subseteq)$-M$(in; 3 \subseteq)$-TS.

Theorem 13. For each universe E, $\langle \mathcal{P}(E^*), \mathcal{C}^*, \cup^*, \square, \cup^* \rangle$ is an IF$\subseteq$-F$(in, \cup^*; 1 \subseteq)$-M$(cl; 3 \supseteq)$-TS.

Theorem 14. For each universe E, $\langle \mathcal{P}(E^*), \mathcal{I}^*, \cap^*, \diamondsuit, \cap^* \rangle$ is an IF$\supseteq$-F$(cl, \cap^*; 1 \supseteq)$-M$(in; 3 \subseteq)$-TS.

Theorem 15. For each universe E, $\langle \mathcal{P}(E^*), \mathcal{C}^*, \cup^*, \square, \cap^* \rangle$ is an IF$\subseteq$-F$(in, \cap^*, 1 \subseteq)$-M$(cl, 3 \supseteq)$-TS.

Theorem 16. For each universe E, $\langle \mathcal{P}(E^*), \mathcal{I}^*, \cap^*, \diamondsuit, \cup^* \rangle$ is an IF$\supseteq$-F$(cl, \cup^*, 1 \supseteq)$-M$(in, 3 \subseteq)$-TS.

Theorem 17. For each universe E, $\langle \mathcal{P}(E^*), \mathcal{C}^*, \cup^*, \diamondsuit, \cap^* \rangle$ is an IF$\supseteq$-F$(cl, \cap^*, 1 \supseteq)$-M$(cl, 3 \supseteq)$-TS.

Theorem 18. For each universe E, $\langle \mathcal{P}(E^*), \mathcal{I}^*, \cap^*, \square, \cup^* \rangle$ is an IF$\subseteq$-F$(in, \cup^*, 1 \subseteq)$-M$(in, 3 \subseteq)$-TS.

Intuitionistic Fuzzy Feeble Modal Topological Structures with operators $\mathcal{C}^{\varepsilon,\eta}$ and $\mathcal{I}^{\varepsilon,\eta}$

In this Section, first, we will construct IFMTSs on the basis of the two topological operations $\mathcal{C}^{\varepsilon,\eta}$ and $\mathcal{I}^{\varepsilon,\eta}$ that are defined by

$$\mathcal{C}^{\varepsilon,\eta}(A) = \{\langle x, \min(1, \sup_{y\in E} \mu_A(y)+\varepsilon), \max(0, \inf_{y\in E} \nu_A(y)-\eta)\rangle | x \in E\}$$

and

$$\mathcal{I}^{\varepsilon,\eta}(A) = \{\langle x, \max(0, \inf_{y\in E} \mu_A(y)-\eta), \min(1, \sup_{y\in E} \nu_A(y)+\varepsilon)\rangle | x \in E\}$$

for every two fixed ε and η, such that $0 \leq \varepsilon \leq \eta \leq 1$, and of the two standard modal operators. After this, we will replace the latter operators with other ones.

Both topological operators are based on operations $\cap^{\varepsilon,\eta}$ and $\cup^{\varepsilon,\eta}$ that have the forms:

$$\begin{aligned} A \cup^{\varepsilon,\eta} B &= \{\langle x, \min(1, \max(\mu_A(x), \mu_B(x)) + \varepsilon), \\ &\qquad \max(0, \min(\nu_A(x), \nu_B(x)) - \eta)\rangle | x \in E\}, \\ A \cap^{\varepsilon,\eta} B &= \{\langle x, \max(0, \min(\mu_A(x), \mu_B(x)) - \eta), \\ &\qquad \min(1, \max(\nu_A(x), \nu_B(x)) + \varepsilon)\rangle | x \in E\}, \end{aligned}$$

where ε and η are described above.

Theorem 19. For each universe E, $\langle \mathcal{P}(E^*), \mathcal{C}^{\varepsilon,\eta}, \cup^{\varepsilon,\eta}, \Box, \cup \rangle$ is an IF$\subseteq$-F$(in, \cup)$-M$(cl, 3 \supseteq, 4 \supseteq)$-TS.

Theorem 20. For each universe E, $\langle \mathcal{P}(E^*), \mathcal{C}^{\varepsilon,\eta}, \cup^{\varepsilon,\eta}, \Box, \cap \rangle$ is an IF$\subseteq$-F$(in, \cap)$-M$(cl, 3 \supseteq, 4 \supseteq)$-TS.

Theorem 21. For each universe E, $\langle \mathcal{P}(E^*), \mathcal{C}^{\varepsilon,\eta}, \cup^{\varepsilon,\eta}, \Diamond, \cap \rangle$ is an IF$\supseteq$-F$(cl, \cap)$-M$(cl, 3 \supseteq, 4 \supseteq)$-TS.

Theorem 22. For each universe E, $\langle \mathcal{P}(E^*), \mathcal{C}^{\varepsilon,\eta}, \cup^{\varepsilon,\eta}, \Diamond, \cup \rangle$ is an IF$\supseteq$-F$(cl, \cup)$-M$(cl, 3 \supseteq, 4 \supseteq)$-TS.

Theorem 23. For each universe E, $\langle \mathcal{P}(E^*), \mathcal{I}^{\varepsilon,\eta}, \cap^{\varepsilon,\eta}, \Box, \cup \rangle$ is an IF$\subseteq$-F$(in, \cup)$-M$(cl, 3 \subseteq, 4 \subseteq)$-TS.

Theorem 24. For each universe E, $\langle \mathcal{P}(E^*), \mathcal{I}^{\varepsilon,\eta}, \cap^{\varepsilon,\eta}, \Box, \cap \rangle$ is an IF$\subseteq$-F$(in, \cap)$-M$(cl, 3 \subseteq, 4 \subseteq)$-TS.

Theorem 25. For each universe E, $\langle \mathcal{P}(E^*), \mathcal{I}^{\varepsilon,\eta}, \cap^{\varepsilon,\eta}, \Diamond, \cup \rangle$ is an IF$\supseteq$-F$(cl, \cup)$-M$(cl, 3 \subseteq, 4 \subseteq)$-TS.

Theorem 26. For each universe E, $\langle \mathcal{P}(E^*), \mathcal{I}^{\varepsilon,\eta}, \cap^{\varepsilon,\eta}, \Diamond, \cap \rangle$ is an IF$\supseteq$-F$(cl, \cap)$-M$(cl, 3 \subseteq, 4 \subseteq)$-TS.

Theorem 27. For each universe E, $\langle\mathcal{P}(E^*), \mathcal{C}^{\varepsilon,\eta}, \cup^{\varepsilon,\eta}, \square, \cup^{\varepsilon,\eta}\rangle$ is an IF$\subseteq$-F$(in, \cup^{\varepsilon,\eta}, 1 \subseteq)$-M$(cl, 3 \supseteq, 4 \supseteq)$-TS.

Theorem 28. For each universe E, $\langle\mathcal{P}(E^*), \mathcal{I}^{\varepsilon,\eta}, \cap^{\varepsilon,\eta}, \square, \cap^{\varepsilon,\eta}\rangle$ is an IF$\subseteq$-F$(in, \cap^{\varepsilon,\eta}, 1 \supseteq)$-M$(cl, 3 \subseteq, 4 \subseteq)$-TS.

Theorem 29. For each universe E, $\langle\mathcal{P}(E^*), \mathcal{I}^{\varepsilon,\eta}, \cap^{\varepsilon,\eta}, \diamondsuit, \cup^{\varepsilon,\eta}\rangle$ is an IF$\supseteq$-F$(cl, \cup^{\varepsilon,\eta} 1 \supseteq)$-M$(cl, 3 \subseteq, 4 \subseteq)$-TS.

Theorem 30. For each universe E, $\langle\mathcal{P}(E^*), \mathcal{I}^{\varepsilon,\eta}, \cap^{\varepsilon,\eta}, \diamondsuit, \cap^{\varepsilon,\eta}\rangle$ is an IF$\supseteq$-F$(cl, \cap^{\varepsilon,\eta} 1 \subseteq)$-M$(cl, 3 \subseteq, 4 \subseteq)$-TS.

Intuitionistic Fuzzy Feeble Modal Topological Structures with operators $\mathcal{C}^{\varepsilon,\eta}$, $\mathcal{I}^{\varepsilon,\eta}$ and $H_{\alpha,0}$, $J_{0,\alpha}$

In the present Section, we will use the following particular forms of the extended modal operators $H_{\alpha,\beta}$ and $J_{\alpha,\beta}$:

$$H_{\alpha,0}(A) = \langle x, \alpha\mu_A(x), \nu_A(x)\rangle | x \in E\},$$

$$J_{0,\alpha}(A) = \langle x, \mu_A(x), \alpha\nu_A(x)\rangle | x \in E\},$$

where A is an IFS and $\alpha \in [0, 1]$.

Theorem 31. For each universe E, $\langle \mathcal{P}(E^*), \mathcal{C}^{\varepsilon,\eta}, \cup, H_{\alpha,0}, \cap\rangle$ is an IF$\subseteq$-F$(in, \cap)$-M(cl)-TS.

Theorem 32. For each universe E, $\langle \mathcal{P}(E^*), \mathcal{C}^{\varepsilon,\eta}, \cup, H_{\alpha,0}, \cup\rangle$ is an IF$\subseteq$-F$(in, \cup)$-M(cl)-TS.

Theorem 33. For each universe E, $\langle \mathcal{P}(E^*), \mathcal{C}^{\varepsilon,\eta}, \cup, J_{0,\alpha}, \cap\rangle$ is an IF$\subseteq$-F$(in, \cap)$-M(cl)-TS.

Theorem 34. For each universe E, $\langle \mathcal{P}(E^*), \mathcal{C}^{\varepsilon,\eta}, \cup, J_{0,\alpha}, \cup\rangle$ is an IF$\subseteq$-F$(in, \cup)$-M(cl)-TS.

Theorem 35. For each universe E, $\langle \mathcal{P}(E^*), \mathcal{I}^{\varepsilon,\eta}, \cap, J_{0,\alpha}, \cap \rangle$ is an IF$\supseteq$-F$(in, \cap)$-M(cl)-TS.

Theorem 36. For each universe E, $\langle \mathcal{P}(E^*), \mathcal{I}^{\varepsilon,\eta}, \cap, J_{0,\alpha}, \cup \rangle$ is an IF$\supseteq$-F$(in, \cap)$-M(cl)-TS.

Theorem 37. For each universe E, $\langle \mathcal{P}(E^*), \mathcal{I}^{\varepsilon,\eta}, \cap, H_{\alpha,0}, \cap \rangle$ is an IF$\supseteq$-F$(in, \cap)$-M(cl)-TS.

Theorem 38. For each universe E, $\langle \mathcal{P}(E^*), \mathcal{I}^{\varepsilon,\eta}, \cap, H_{\alpha,0}, \cup \rangle$ is an IF$\supseteq$-F$(in, \cup)$-M(cl)-TS.

Conclusion

In the present paper, we gave unified definitions for the different types of FMTSs and re-formulated the results from the previous ones devoted to IFFMTSs. In future, we will discuss some other types of MTSs, illustrating them with temporal IFSs, IFSs over which level-operators are defined, and others.

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