

# **Multi-dimensional modal topological structures, extended intuitionistic fuzzy index matrices and multi-person multi-criteria decision making procedures**

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# Introduction

The paper is a continuation of previous author's research. In it, examples of Modal Topological Structures (MTSs) with elements from the area of intuitionistic fuzziness are described. Here, new examples related to the extension of the MTSs called Multi-Dimensional MTSs (MDMTSs) are introduced.

In Section 2 we give short remarks on MDMTSs, Intuitionistic Fuzzy MDMTSs (IFMDMTSs) and Extended Intuitionistic Fuzzy IMs (EIFIMs). In Section 3 we give 16 IFMDMTSs based on EIFIMs. Finally, in the Conclusion we discuss some other possible applications of the IFMDMTSs.

## Preliminaries

Below, we will give some definitions related to MDMTSs and EIFIMs.

### **Definition of the Multi-Dimensional Modal Topological Structure**

First, we will define the concept of a MDMTS, or, when the number of the dimensions is determined, e.g., as  $s \geq 1$  for some natural number  $s$  – as  $s$ -dimensional MTS ( $s$ -DMTS).

Let for the arbitrary set  $Z$  we define

$$\mathcal{P}(Z) = \{Y \mid Y \subseteq Z\}.$$

Let us have the fixed sets  $X_1, \dots, X_s$ , where the natural number  $s \geq 1$ .

Let

$$\langle \mathcal{P}(X_1), \mathcal{E}, \zeta, *, \eta \rangle,$$

...

$$\langle \mathcal{P}(X_s), \mathcal{E}, \zeta, *, \eta \rangle$$

be given, where for each  $r$  ( $1 \leq r \leq s$ ), the object  $\langle \mathcal{P}(X_r), \mathcal{E}, \zeta, *, \eta \rangle$  is a MTS.

Now, following the definition of a topological structure and the definition for a MTS, we construct the object

$$\langle \mathcal{P}(X_1 \times \cdots \times X_s), \mathcal{E}, \zeta_s, *, \eta_s \rangle$$

that will denote as an  $s$ -Dimensional  $(\chi, \eta_s)$ -Modal  $(\varphi, \zeta_s)$ -Topological Structure ( $s$ -D  $(\chi, \eta_s)$ -M  $(\varphi, \zeta_s)$ -TS) or (more general) multi-Dimensional  $(\chi, \eta_s)$ -Modal  $(\varphi, \zeta_s)$ -Topological Structure (mD  $(\chi, \eta_s)$ -M  $(\varphi, \zeta_s)$ -TS), where for  $A, B \in \mathcal{P}(X_1 \times \cdots \times X_s)$ :

- $\zeta_s : (X_1 \times \cdots \times X_s) \times (X_1 \times \cdots \times X_s) \rightarrow X_1 \times \cdots \times X_s$  is an associative operation, being a generation of the function  $\zeta$  in the following, for example, sense: if

$$\zeta_2(a_1, a_2) = \zeta(a_1, a_2) = a_1 a_2 = \prod_{i=1}^2 a_i,$$

then

$$\zeta_s(a_1, \dots, a_s) = \prod_{i=1}^s a_i;$$

- $\mathcal{E}$  is a topological operator and if it is from closure (*cl*) type, then it must satisfy the conditions

Ct1  $\mathcal{E}(A\zeta_s B) = \mathcal{E}(A)\zeta_s \mathcal{E}(B),$

Ct2  $A \subseteq \mathcal{E}(A),$

Ct3  $\mathcal{E}(\mathcal{E}(A)) = \mathcal{E}(A),$

Ct4  $\mathcal{E}(O_1 \times \cdots \times O_s) = O_1 \times \cdots \times O_s,$

where  $O_r$  is the minimal element of  $\mathcal{P}(X_r)$  for  $1 \leq r \leq s$ ,

$O = O_1 \times \cdots \times O_s$  is the minimal element of the set

$\mathcal{P}(X_1 \times \cdots \times X_s)$ , and if it is from interior (*in*) type, then the topological operator  $\mathcal{E}$  must satisfy the conditions

**It1**  $\mathcal{E}(A\zeta_s B) = \mathcal{E}(A)\zeta_s \mathcal{E}(B)$ ,

**It2**  $\mathcal{E}(A) \subseteq A$ ,

**It3**  $\mathcal{E}(\mathcal{E}(A)) = \mathcal{E}(A)$ ,

**It4**  $\mathcal{E}(X_1 \times \cdots \times X_s) = X_1 \times \cdots \times X_s$ ;

- $\eta_s : (X_1 \times \cdots \times X_s) \times (X_1 \times \cdots \times X_s) \rightarrow X_1 \times \cdots \times X_s$  is an associative operation being a generation of the function  $\eta$  in the above, for example, sense: if

$$\eta_2(a_1, a_2) = \eta(a_1, a_2) = \min(a_1, a_2) = \min_{1 \leq i \leq 2} a_i,$$

then

$$\eta_s(a_1, \dots, a_s) = \min_{1 \leq i \leq s} a_i;$$

- \* is a modal operator and if it is from closure (*cl*) type, then it must satisfy the conditions

**Cm1**  $*(A\eta_s B) = *A\eta_s *B,$

**Cm2**  $A \subseteq *A,$

**Cm3**  $* * A = *A,$

and if it is from interior (*in*) type, then it must satisfy the conditions

**Im1**  $*(A\eta_s B) = *A\eta_s *B,$

**Im2**  $*A \subseteq A,$

**Im3**  $* * A = *A;$

- $\chi, \varphi \in \{cl, in\}$ ,
- both operators (topological and modal) must satisfy the condition

$$*\mathcal{E}(A) = \mathcal{E}(*A). \quad (*)$$

We can see immediately that when  $s = 1$ ,  $\zeta_2 = \zeta$ ,  $\eta_2 = \eta$ , we obtain the definition of a MTS.

If we add as an additional conditions the operations  $\zeta_s$  and  $\eta_s$  to be commutative and for every  $i, j$  ( $1 \leq i < j \leq s$ ) and for each  $A_1 \times \cdots \times A_s \in \mathcal{P}(X_1 \times \cdots \times X_s)$ :

$$\mathcal{E}(A_1 \times \cdots \times A_i \times \cdots \times A_j \times \cdots \times A_s) = \mathcal{E}(A_1 \times \cdots \times A_j \times \cdots \times A_i \times \cdots \times A_s), \quad (**)$$

then the MDMTS will be called a commutative MDMTS.

When  $E = E_1 \times \cdots \times E_s$ , where  $E_1, \dots, E_s$  are fixed universes, and for each its subset  $A$  we can construct the IFS (more exactly - multi- (or  $s$ -)dimensional IFS

$$A^* = \{\langle x, \mu_A(x_1, \dots, x_s), \nu_A(x_1, \dots, x_s) \rangle \mid \langle x_1, \dots, x_s \rangle \in E\},$$

then

$$\langle \mathcal{P}(E_1 \times \cdots \times E_s), \mathcal{E}, \zeta_s, *, \eta_s \rangle$$

is an IFMDMTS.

## Short remarks on EIFIMs

Let  $I$  be a fixed set. By EIFIM with index sets  $K$  and  $L$  ( $K, L \subset I$ ), we denote the object:

$$[K^*, L^*, \{\langle \mu_{k_i, l_j}, \nu_{k_i, l_j} \rangle\}] \equiv$$

	$l_1, \langle \alpha_1^L, \beta_1^L \rangle$	$\dots$	$l_j, \langle \alpha_j^L, \beta_j^L \rangle$	$\dots$	$l_n, \langle \alpha_n^L, \beta_n^L \rangle$
$k_1, \langle \alpha_1^K, \beta_1^K \rangle$	$\langle \mu_{k_1, l_1}, \nu_{k_1, l_1} \rangle$	$\dots$	$\langle \mu_{k_1, l_j}, \nu_{k_1, l_j} \rangle$	$\dots$	$\langle \mu_{k_1, l_n}, \nu_{k_1, l_n} \rangle$
$\vdots$	$\vdots$	$\dots$	$\vdots$	$\dots$	$\vdots$
$k_i, \langle \alpha_i^K, \beta_i^K \rangle$	$\langle \mu_{k_i, l_1}, \nu_{k_i, l_1} \rangle$	$\dots$	$\langle \mu_{k_i, l_j}, \nu_{k_i, l_j} \rangle$	$\dots$	$\langle \mu_{k_i, l_n}, \nu_{k_i, l_n} \rangle$
$\vdots$	$\vdots$	$\dots$	$\vdots$	$\dots$	$\vdots$
$k_m, \langle \alpha_m^K, \beta_m^K \rangle$	$\langle \mu_{k_m, l_1}, \nu_{k_m, l_1} \rangle$	$\dots$	$\langle \mu_{k_m, l_j}, \nu_{k_m, l_j} \rangle$	$\dots$	$\langle \mu_{k_m, l_n}, \nu_{k_m, l_n} \rangle$

where

$$K = \{k_1, k_2, \dots, k_m\},$$

$$L = \{l_1, l_2, \dots, l_n\};$$

$$K^* = \{\langle k_i, \alpha_i^K, \beta_i^K \rangle | k_i \in K\} = \{\langle k_i, \alpha_i^K, \beta_i^K \rangle | 1 \leq i \leq m\},$$

$$L^* = \{\langle l_j, \alpha_j^L, \beta_j^L \rangle | l_j \in L\} = \{\langle l_j, \alpha_j^L, \beta_j^L \rangle | 1 \leq j \leq n\}$$

are IFSs and for every  $1 \leq i \leq m, 1 \leq j \leq n$ :

$$\mu_{k_i, l_j}, \nu_{k_i, l_j}, \mu_{k_i, l_j} + \nu_{k_i, l_j} \in [0, 1],$$

$$\alpha_i^K, \beta_i^K, \alpha_i^K + \beta_i^K \in [0, 1],$$

$$\alpha_j^L, \beta_j^L, \alpha_j^L + \beta_j^L \in [0, 1].$$

For the EIFIMs

$$A = [K^*, L^*, \{\langle \mu_{k_i, l_j}, \nu_{k_i, l_j} \rangle\}],$$

$$B = [P^*, Q^*, \{\langle \rho_{p_r, q_s}, \sigma_{p_r, q_s} \rangle\}],$$

we will introduce only these operations, relations and operators that are necessary for the current research.

## Addition-(max,min)

$$A \oplus_{(\max, \min)} B = [T^*, V^*, \{\langle \varphi_{t_u, v_w}, \psi_{t_u, v_w} \rangle\}],$$

where

$$T^* = K^* \cup P^* = \{\langle t_u, \langle \alpha_u^K, \beta_u^K \rangle \vee \langle \alpha_u^P, \beta_u^P \rangle \rangle | t_u \in K \cup P\},$$

$$= \{\langle t_u, \alpha_u^T, \beta_u^T \rangle | t_u \in K \cup P\}$$

$$V^* = L^* \cup Q^* = \{\langle v_w, \langle \alpha_w^L, \beta_w^L \rangle \vee \langle \alpha_w^Q, \beta_w^Q \rangle \rangle | v_w \in L \cup Q\},$$

$$= \{\langle v_w, \alpha_w^V, \beta_w^V \rangle | v_w \in L \cup Q\},$$

where

$$\alpha_u^T = \begin{cases} \alpha_i^K, & \text{if } t_u \in K - P \\ \alpha_r^P, & \text{if } t_u \in P - K \\ \max(\alpha_i^K, \alpha_r^P), & \text{if } t_u \in K \cap P \end{cases}$$
$$\beta_w^V = \begin{cases} \beta_j^L, & \text{if } v_w \in L - Q \\ \beta_s^Q, & \text{if } t_w \in Q - L \\ \min(\beta_j^L, \beta_s^Q), & \text{if } t_w \in L \cap Q \end{cases}$$

and

$$\langle \varphi_{t_u, v_w}, \psi_{t_u, v_w} \rangle = \begin{cases} \langle \mu_{k_i, l_j}, \nu_{k_i, l_j} \rangle, & \begin{array}{l} \text{if } t_u = k_i \in K \\ \text{and } v_w = l_j \in L - Q \\ \text{or } t_u = k_i \in K - P \\ \text{and } v_w = l_j \in L; \end{array} \\ \langle \rho_{p_r, q_s}, \sigma_{p_r, q_s} \rangle, & \begin{array}{l} \text{if } t_u = p_r \in P \\ \text{and } v_w = q_s \in Q - L \\ \text{or } t_u = p_r \in P - K \\ \text{and } v_w = q_s \in Q; \end{array} \\ \langle \max(\mu_{k_i, l_j}, \rho_{p_r, q_s}), \\ \min(\nu_{k_i, l_j}, \sigma_{p_r, q_s}) \rangle, & \begin{array}{l} \text{if } t_u = k_i = p_r \in K \cap P \\ \text{and } v_w = l_j = q_s \in L \cap Q \end{array} \\ \langle 0, 1 \rangle, & \text{otherwise} \end{cases}$$

## Addition-(min,max)

$$A \oplus_{(\min, \max)} B = [T^*, V^*, \{\langle \varphi_{t_u, v_w}, \psi_{t_u, v_w} \rangle\}],$$

where  $T^*, V^*, \alpha_u^T, \beta_w^V$ , have the above forms and

$$\langle \varphi_{t_u, v_w}, \psi_{t_u, v_w} \rangle = \begin{cases} \langle \mu_{k_i, l_j}, \nu_{k_i, l_j} \rangle, & \begin{array}{l} \text{if } t_u = k_i \in K \\ \text{and } v_w = l_j \in L - Q \\ \text{or } t_u = k_i \in K - P \\ \text{and } v_w = l_j \in L; \end{array} \\ \langle \rho_{p_r, q_s}, \sigma_{p_r, q_s} \rangle, & \begin{array}{l} \text{if } t_u = p_r \in P \\ \text{and } v_w = q_s \in Q - L \\ \text{or } t_u = p_r \in P - K \\ \text{and } v_w = q_s \in Q; \end{array} \\ \langle \min(\mu_{k_i, l_j}, \rho_{p_r, q_s}), \\ \max(\nu_{k_i, l_j}, \sigma_{p_r, q_s}) \rangle, & \begin{array}{l} \text{if } t_u = k_i = p_r \in K \cap P \\ \text{and } v_w = l_j = q_s \in L \cap Q \end{array} \\ \langle 0, 1 \rangle, & \text{otherwise} \end{cases}$$

## Termwise multiplication-(max,min)

$$A \otimes_{(\max, \min)} B = [T^*, V^*, \{\langle \varphi_{t_u, v_w}, \psi_{t_u, v_w} \rangle\}],$$

where

$$T^* = K^* \cap P^* = \{\langle t_u, \langle \alpha_u^K, \beta_u^K \rangle \& \langle \alpha_u^P, \beta_u^P \rangle \rangle | t_u \in K \cup P\},$$

$$= \{\langle t_u, \alpha_u^T, \beta_u^T \rangle | t_u \in K \cup P\}$$

$$V^* = L^* \cap Q^* = \{\langle v_w, \langle \alpha_w^L, \beta_w^L \rangle \& \langle \alpha_w^Q, \beta_w^Q \rangle \rangle | v_w \in L \cup Q\},$$

$$= \{\langle v_w, \alpha_w^V, \beta_w^V \rangle | v_w \in L \cup Q\},$$

$$\alpha_u^T = \min(\alpha_i^K, \alpha_r^P), \text{ for } t_u = k_i = p_r \in K \cap P,$$

$$\beta_w^V = \min(\beta_j^L, \beta_s^Q), \text{ for } v_w = l_j = q_s \in L \cap Q$$

and

$$\langle \varphi_{t_u, v_w}, \psi_{t_u, v_w} \rangle = \langle \max(\mu_{k_i, l_j}, \rho_{p_r, q_s}), \min(\nu_{k_i, l_j}, \sigma_{p_r, q_s}) \rangle.$$

## Termwise multiplication-(min,max)

$$A \otimes_{(\min, \max)} B = [T^*, V^*, \{\langle \varphi_{t_u, v_w}, \psi_{t_u, v_w} \rangle\}],$$

where  $T^*, V^*, \alpha_u^T, \beta_w^V$ , have the above forms and

$$\langle \varphi_{t_u, v_w}, \psi_{t_u, v_w} \rangle = \langle \min(\mu_{k_i, l_j}, \rho_{p_r, q_s}), \max(\nu_{k_i, l_j}, \sigma_{p_r, q_s}) \rangle.$$

Let

$$K^* \subset P^* \text{ iff } (K \subset P) \ \& \ (\forall k_i = p_i \in K)((\alpha_i^K < \alpha_i^P) \ \& \ (\beta_i^K > \beta_i^P)),$$

$$K^* \subseteq P^* \text{ iff } (K \subseteq P) \ \& \ (\forall k_i = p_i \in K)((\alpha_i^K \leq \alpha_i^P) \ \& \ (\beta_i^K \geq \beta_i^P)).$$

Let the two EIFIMs  $A$  and  $B$  be given. We shall introduce the following definitions for relations over EIFIMs, where  $\subset$  and  $\subseteq$  denote the relations “*strong inclusion*” and “*weak inclusion*”, respectively. We must mention that in the present form, they are new and extend existing definitions given in [?].

**The strict relation “inclusion about dimension” is**

$$\begin{aligned}
 A \subset_d B \quad \text{iff } & (((K^* \subset P^*) \ \& \ (L^* \subset Q^*)) \ \vee \ ((K^* \subseteq P^*) \ \& \ (L^* \subset Q^*)) \\
 & \vee ((K^* \subset P^*) \ \& \ (L^* \subseteq Q^*))) \ \& \ (\forall k \in K)(\forall l \in L)((\langle \alpha_k^K, \beta_k^K \rangle \\
 & \& (\langle \alpha_l^L, \beta_l^L \rangle = \langle \alpha_l^Q, \beta_l^Q \rangle) \ \& \ (\langle \mu_{k,l}, \nu_{k,l} \rangle = \langle \rho_{k,l}, \sigma_{k,l} \rangle)).
 \end{aligned}$$

**The non-strict relation “inclusion about dimension” is**

$$\begin{aligned} A \subseteq_d B \quad \text{iff } & (K^* \subseteq P^*) \ \& \ (L^* \subseteq Q^*) \ \& \ (\forall k \in K)(\forall l \in L) \\ & ((\langle \alpha_k^K, \beta_k^K \rangle = \langle \alpha_k^P, \beta_k^P \rangle) \ \& \ (\langle \alpha_l^L, \beta_l^L \rangle = \langle \alpha_l^Q, \beta_l^Q \rangle)) \\ & \ \& \ (\langle \mu_{k,l}, \nu_{k,l} \rangle = \langle \rho_{k,l}, \sigma_{k,l} \rangle)). \end{aligned}$$

**The strict relation “inclusion about value” is**

$$\begin{aligned} A \subset_v B \quad \text{iff } & (K^* = P^*) \ \& \ (L^* = Q^*) \ \& \ (\forall k \in K)(\forall l \in L) \\ & ((\langle \alpha_k^K, \beta_k^K \rangle = \langle \alpha_k^P, \beta_k^P \rangle) \ \& \ (\langle \alpha_l^L, \beta_l^L \rangle = \langle \alpha_l^Q, \beta_l^Q \rangle)) \\ & \ \& \ (\langle \mu_{k,l}, \nu_{k,l} \rangle < \langle \rho_{k,l}, \sigma_{k,l} \rangle)). \end{aligned}$$

**The non-strict relation “inclusion about value” is**

$$A \subseteq_v B \quad \text{iff } (K^* = P^*) \ \& \ (L^* = Q^*) \ \& \ (\forall k \in K)(\forall l \in L) \\ ((\langle \alpha_k^K, \beta_k^K \rangle = \langle \alpha_k^P, \beta_k^P \rangle) \ \& \ (\langle \mu_{k,l}, \nu_{k,l} \rangle \leq \langle \rho_{k,l}, \sigma_{k,l} \rangle)).$$

**The strict relation “inclusion about indices” is**

$$A \subset_i B \quad \text{iff } (K^* = P^*) \ \& \ (L^* = Q^*) \ \& \ (\forall k \in K)(\forall l \in L) \\ ((\langle \alpha_k^K, \beta_k^K \rangle < \langle \alpha_k^P, \beta_k^P \rangle) \ \& \ (\langle \alpha_l^L, \beta_l^L \rangle < \langle \alpha_l^Q, \beta_l^Q \rangle) \\ \& (\langle \mu_{k,l}, \nu_{k,l} \rangle = \langle \rho_{k,l}, \sigma_{k,l} \rangle)).$$

**The non-strict relation “inclusion about indices” is**

$$\begin{aligned} A \subseteq_i B \quad \text{iff } & (K^* = P^*) \ \& \ (L^* = Q^*) \ \& \ (\forall k \in K)(\forall l \in L) \\ & ((\langle \alpha_k^K, \beta_k^K \rangle \leq \langle \alpha_k^P, \beta_k^P \rangle) \ \& \ (\langle \alpha_l^L, \beta_l^L \rangle \leq \langle \alpha_l^Q, \beta_l^Q \rangle) \\ & \ \& \ (\langle \mu_{k,l}, \nu_{k,l} \rangle = \langle \rho_{k,l}, \sigma_{k,l} \rangle)). \end{aligned}$$

Now, we can introduce combinations of these relations in the forms

$$A \subset_{d,v} B \quad \text{iff } (A \subset_d B) \ \& \ (A \subset_v B),$$

$$A \subseteq_{d,v} B \quad \text{iff } (A \subseteq_d B) \ \& \ (A \subseteq_v B),$$

$$A \subset_{d,i} B \quad \text{iff } (A \subset_d B) \ \& \ (A \subset_i B),$$

$$A \subseteq_{d,i} B \quad \text{iff } (A \subseteq_d B) \ \& \ (A \subseteq_i B),$$

$$A \subset_{v,i} B \quad \text{iff } (A \subset_v B) \ \& \ (A \subseteq_i B),$$

$$A \subseteq_{v,i} B \quad \text{iff } (A \subseteq_v B) \ \& \ (A \subseteq_i B),$$

$$A \subset B \quad \text{iff } (A \subset_d B) \ \& \ (A \subset_v B) \ \& \ (A \subset_i B),$$

$$A \subseteq B \quad \text{iff } (A \subseteq_d B) \ \& \ (A \subseteq_v B) \ \& \ (A \subseteq_i B).$$

## Extended intuitionistic fuzzy index matrices as multi-dimensional modal topological structures

Let us have two sets of indices  $I_1$  and  $I_2$  and a universe  $E$ . Let

$$I_1^* = \{\langle i_1, 1, 0 \rangle | i_1 \in I_1\},$$

$$I_2^* = \{\langle i_2, 1, 0 \rangle | i_2 \in I_2\},$$

$$E^* = \{I_1^*, I_2^*, \langle x, 1, 0 \rangle | x \in E\},$$

$$O^* = \{I_1^*, I_2^*, \langle x, 0, 1 \rangle | x \in E\}.$$

These sets are IFSs and for them, similarly to [?], we can construct the IFMTSs

$$\langle \mathcal{P}(I_1^*), \mathcal{E}, \zeta, *, \eta \rangle,$$

$$\langle \mathcal{P}(I_2^*), \mathcal{E}, \zeta, *, \eta \rangle,$$

$$\langle \mathcal{P}(E^*), \mathcal{E}, \zeta, *, \eta \rangle,$$

where  $\mathcal{E} \in \{\mathcal{C}, \mathcal{I}\}$ ,  $* \in \{\square, \diamond\}$ ,  $\zeta$  and  $\eta$  are functions and  $\mathcal{C}, \mathcal{I}, \square, \diamond$  are the standard intuitionistic fuzzy topological (from closure and interior type) and modal (from necessity and possibility type) operators, respectively.

Therefore, we can construct the object

$$\langle \mathcal{P}(I_1^* \times I_2^2 \times E^*), \mathcal{E}, \zeta_3, *, \eta_3 \rangle.$$

The elements of the set  $\mathcal{P}(I_1^* \times I_2^2 \times E^*)$  are IFSs with the form

$$A^* = \{ \langle \langle \langle k_i, \alpha_i^K, \beta_i^K \rangle, \langle l_i, \alpha_i^L, \beta_i^L \rangle \rangle, \mu(k_i, l_i), \nu(k_i, l_i) \rangle \mid \langle k_i, l_i \rangle \in K \times L \},$$

where  $K^* \subseteq \mathcal{I}_1^*$  and  $L^* \subseteq \mathcal{I}_2^*$ .

Now, we can construct the EIFIM

$$A = [K^*, L^*, \{ \langle \mu(k_i, l_i), \nu(k_i, l_i) \rangle \mid k_i \in K, l_i \in L \}].$$

It is interesting to mention that in the previous research we perceived the set  $I_1 \times I_2$  as one set, while below we will perceive the object  $I_1^* \times I_2^2 \times E^*$  as a Cartesian product of three sets.

Obviously, the EIFIM  $A$  is another form of the IFS  $A^*$ . Therefore, on it we can define topological and modal operators as follows:

$$\mathcal{C}(A) = [\{\langle k_i, \sup_{k \in K} \alpha_k^K, \inf_{k \in K} \beta_k^K \rangle | k_i \in K\}, \{\langle l_j, \sup_{l \in L} \alpha_l^L, \inf_{l \in L} \beta_l^L \rangle | l_j \in L\},$$

$$\{\langle \sup_{\langle k, l \rangle \in K \times L} \mu(k, l), \inf_{\langle k, l \rangle \in K \times L} \nu(k, l) \rangle | k_i \in K, l_j \in L\}],$$

$$\mathcal{I}(A) = [\{\langle k_i, \inf_{k \in K} \alpha_k^K, \sup_{k \in K} \beta_k^K \rangle | k_i \in K\}, \{\langle l_j, \inf_{l \in L} \alpha_l^L, \sup_{l \in L} \beta_l^L \rangle | l_j \in L\},$$

$$\{\langle \inf_{\langle k, l \rangle \in K \times L} \mu(k, l), \sup_{\langle k, l \rangle \in K \times L} \nu(k, l) \rangle | k_i \in K, l_j \in L\}],$$

$$\square A = [\{\langle k, \alpha_k^K, 1 - \alpha_k^K \rangle | k \in K\}, \{\langle l, \alpha_l^L, 1 - \alpha_l^L \rangle | l \in L\}, \\ \{\langle \mu(k, l), 1 - \mu(k, l) \rangle\} | k \in K, l \in L\}],$$

$$\diamond A = [\{\langle k, 1 - \beta_k^K, \beta_k^K \rangle | k \in K\}, \{\langle l, 1 - \beta_l^L, \beta_l^L \rangle | l \in L\}, \\ \{\langle \mu(k, l), 1 - \mu(k, l) \rangle\} | k \in K, l \in L\}].$$

It is important to mention that in the previous research we perceived the set  $I_1 \times I_2$  as one set, while below we will perceive the object  $I_1^* \times I_2^2 \times E^*$  as a Cartesian product of three sets. In addition, these sets are such that just as a topological or modal operator can be defined over each of them separately, the same can be done over their product. Therefore, in the sense of the definition from Section 2.1, the structure will be 3-dimensional one.

Now, we can prove the following assertion.

**Theorem 1.** For every three IFSs  $I_1^*, I_2^2, E^*$ :

- (1)  $\langle \mathcal{P}(I_1^* \times I_2^* \times E^*), \mathcal{C}, \oplus_{\vee}, \diamond, \oplus_{\vee} \rangle$  is an IF3DMTS,
- (2)  $\langle \mathcal{P}(I_1^* \times I_2^* \times E^*), \mathcal{C}, \oplus_{\vee}, \diamond, \oplus_{\wedge} \rangle$  is an IF3DMTS,
- (3)  $\langle \mathcal{P}(I_1^* \times I_2^* \times E^*), \mathcal{C}, \oplus_{\vee}, \square, \oplus_{\vee} \rangle$  is an IF3DMTS,
- (4)  $\langle \mathcal{P}(I_1^* \times I_2^* \times E^*), \mathcal{C}, \oplus_{\vee}, \square, \oplus_{\wedge} \rangle$  is an IF3DMTS,
- (5)  $\langle \mathcal{P}(I_1^* \times I_2^* \times E^*), \mathcal{C}, \otimes_{\vee}, \diamond, \oplus_{\vee} \rangle$  is an IF3DMTS,
- (6)  $\langle \mathcal{P}(I_1^* \times I_2^* \times E^*), \mathcal{C}, \otimes_{\vee}, \diamond, \oplus_{\wedge} \rangle$  is an IF3DMTS,
- (7)  $\langle \mathcal{P}(I_1^* \times I_2^* \times E^*), \mathcal{C}, \otimes_{\vee}, \square, \oplus_{\vee} \rangle$  is an IF3DMTS,
- (8)  $\langle \mathcal{P}(I_1^* \times I_2^* \times E^*), \mathcal{C}, \otimes_{\vee}, \square, \oplus_{\wedge} \rangle$  is an IF3DMTS,

- (9)  $\langle \mathcal{P}(I_1^* \times I_2^* \times E^*), \mathcal{I}, \oplus_{\wedge}, \diamondsuit, \oplus_{\vee} \rangle$  is an IF3DMTS,
- (10)  $\langle \mathcal{P}(I_1^* \times I_2^* \times E^*), \mathcal{I}, \oplus_{\wedge}, \diamondsuit, \oplus_{\wedge} \rangle$  is an IF3DMTS,
- (11)  $\langle \mathcal{P}(I_1^* \times I_2^* \times E^*), \mathcal{I}, \oplus_{\wedge}, \square, \oplus_{\vee} \rangle$  is an IF3DMTS,
- (12)  $\langle \mathcal{P}(I_1^* \times I_2^* \times E^*), \mathcal{I}, \oplus_{\wedge}, \square, \oplus_{\wedge} \rangle$  is an IF3DMTS,
- (13)  $\langle \mathcal{P}(I_1^* \times I_2^* \times E^*), \mathcal{I}, \otimes_{\wedge}, \diamondsuit, \oplus_{\vee} \rangle$  is an IF3DMTS,
- (14)  $\langle \mathcal{P}(I_1^* \times I_2^* \times E^*), \mathcal{I}, \otimes_{\wedge}, \diamondsuit, \oplus_{\wedge} \rangle$  is an IF3DMTS,
- (15)  $\langle \mathcal{P}(I_1^* \times I_2^* \times E^*), \mathcal{I}, \otimes_{\wedge}, \square, \oplus_{\vee} \rangle$  is an IF3DMTS,
- (16)  $\langle \mathcal{P}(I_1^* \times I_2^* \times E^*), \mathcal{I}, \otimes_{\wedge}, \square, \oplus_{\wedge} \rangle$  is an IF3DMTS.

## Proof.

Let the sets  $I_1^*$ ,  $I_2^2$  and  $E^*$  be given and let

$$A = [K^*, L^*, \{\langle \mu(k, l), \nu(k, l) \rangle \mid k \in K, l \in L\}] \in \mathcal{P}(I_1^* \times I_2^2 \times E^*),$$

$$B = [P^*, Q^*, \{\langle \mu(p, q), \nu(p, q) \rangle \mid p \in P, q \in Q\}] \in \mathcal{P}(I_1^* \times I_2^2 \times E^*).$$

We must mention that objects  $\{\langle \mu(k, l), \nu(k, l) \rangle \mid k \in K, l \in L\}$  and  $\{\langle \mu(p, q), \nu(p, q) \rangle \mid p \in P, q \in Q\}$  that are the elements of EIFIMs can be interpreted as IFSs. Then for example for (1) we can check the conditions that the object is an EIFMTS as follows

**Ct1** For every two IFSs  $X$  and  $Y$ :  $\mathcal{C}(X \cup Y) = \mathcal{C}(X) \cup \mathcal{C}(Y)$  and from this equality it follows:

$$\begin{aligned}
 \mathcal{C}(A \oplus_{\vee} B) &= \mathcal{C}([K^* \cup P^*, L^* \cup Q^*, \{\langle \mu(k, l), \nu(k, l) \vee \langle \mu(p, q), \nu(p, q) \rangle \mid k, p \in K \cup P; l, q \in L \cup Q\}]) \\
 &= [\mathcal{C}(K^* \cup P^*), \mathcal{C}(L^* \cup Q^*), \{\langle \sup_{k, p \in K \cup P, l, q \in L \cup Q} \max(\mu(k, l), \mu(p, q)), \\
 &\quad \inf_{k, p \in K \cup P, l, q \in L \cup Q} \min(\nu(k, l), \nu(p, q)) \rangle \mid k, p \in K \cup P; l, q \in L \cup Q\}] \\
 &= [\mathcal{C}(K^*) \cup \mathcal{C}(P^*), \mathcal{C}(L^*) \cup \mathcal{C}(Q^*), \{\langle \max(\sup_{\langle k, l \rangle \in K \times L} \mu(k, l), \sup_{\langle p, q \rangle \in P \times Q} \mu(p, q)), \\
 &\quad \min(\inf_{\langle k, l \rangle \in K \times L} \nu(k, l), \inf_{\langle p, q \rangle \in P \times Q} \nu(p, q)) \rangle \mid k \in K, l \in L\}] \\
 &= [\{\mathcal{C}(K^*), \mathcal{C}(L^*), \{\langle \sup_{\langle k, l \rangle \in K \times L} \mu(k, l), \inf_{\langle k, l \rangle \in K \times L} \nu(k, l) \rangle \mid k \in K, l \in L\}] \\
 &\quad \oplus_{\vee} [\mathcal{C}(P^*), \mathcal{C}(Q^*), \{\langle \sup_{\langle p, q \rangle \in P \times Q} \mu(p, q), \inf_{\langle p, q \rangle \in P \times Q} \nu(p, q) \rangle \mid p \in P, q \in Q\}] \\
 &= \mathcal{C}(A) \oplus_{\vee} \mathcal{C}(B);
 \end{aligned}$$

**Ct2** For each IFS  $X$ :  $\mathcal{C}(\mathcal{C}(X)) = \mathcal{C}(X)$  and from this equality it follows:

$$\begin{aligned}
 & \mathcal{C}(\mathcal{C}(A)) \\
 &= \mathcal{C}([\mathcal{C}(K^*), \mathcal{C}(L^*), \{\langle \sup_{\langle k,l \rangle \in K \times L} \mu(k, l), \inf_{\langle k,l \rangle \in K \times L} \nu(k, l) \rangle \mid k \in K, l \in L\}]) \\
 &= [\mathcal{C}(\mathcal{C}(K^*)), \mathcal{C}(\mathcal{C}(L^*)), \{\langle \sup_{\langle p,q \rangle \in K \times L} \sup_{\langle k,l \rangle \in K \times L} \mu(k, l), \\
 & \quad \inf_{\langle p,q \rangle \in K \times L} \inf_{\langle k,l \rangle \in K \times L} \nu(k, l) \rangle \mid k \in K, l \in L\}] \\
 &= [\mathcal{C}(K^*), \mathcal{C}(L^*), \{\langle \sup_{\langle k,l \rangle \in K \times L} \mu(k, l), \inf_{\langle k,l \rangle \in K \times L} \nu(k, l) \rangle \mid k \in K, l \in L\}] \\
 &= \mathcal{C}(A);
 \end{aligned}$$

**Ct3**

$$\begin{aligned}
 A &= [K^*, L^*, \{\langle \mu(k, l), \nu(k, l) \rangle \mid k \in K, l \in L\}] \\
 &\subseteq_{v,i} [\mathcal{C}(K^*), \mathcal{C}(L^*), \{\langle \sup_{\langle k,l \rangle \in K \times L} \mu(k, l), \inf_{\langle k,l \rangle \in K \times L} \nu(k, l) \rangle \mid k \in K, l \in L\}] \\
 &= \mathcal{C}(A);
 \end{aligned}$$

#### Ct4

$$\begin{aligned}
 \mathcal{C}(O^*) &= [\mathcal{C}(I_1^*), \mathcal{C}(I_2^*), \{\langle \sup_{\langle k,l \rangle \in K \times L} 0, \inf_{\langle k,l \rangle \in K \times L} 1 \rangle \mid k \in K, l \in L\}] \\
 &= [I_1^*, I_2^*, \{\langle 0, 1 \rangle \mid k \in K, l \in L\}] \\
 &= O^*;
 \end{aligned}$$

**Cm1** For every two IFSs  $X$  and  $Y$ :  $\diamond(X \cup Y) = \diamond X \cup \diamond Y$  and from this equality it follows:

$$\begin{aligned}
 &\diamond(A \oplus_V B) \\
 &= [\diamond(K^* \cup P^*), \diamond(L^* \cup Q^*), \diamond\{\langle \langle \max(\mu(k, l), \mu(p, q)), \min(\nu(k, l), \nu(p, q)) \rangle \rangle \mid \\
 &\quad k, p \in K \cup P; l, q \in L \cup Q\}] \\
 &= [\diamond K^* \cup \diamond P^*, \diamond L^* \cup \diamond Q^*, \{\langle 1 - \min(\nu(k, l), \nu(p, q)), \min(\nu(k, l), \nu(p, q)) \rangle \mid \\
 &\quad k, p \in K \cup P; l, q \in L \cup Q\}] \\
 &= [\diamond K^* \cup \diamond P^*, \diamond L^* \cup \diamond Q^*, \{\langle \max(1 - \nu(k, l), 1 - \nu(p, q)), \min(\nu(k, l), \nu(p, q)) \rangle \mid \\
 &\quad k, p \in K \cup P; l, q \in L \cup Q\}] \\
 &= [K^*, L^*, \{\langle 1 - \nu(k, l), \nu(k, l) \rangle \mid k \in K, l \in L\}] \\
 &\quad \oplus_V [P^*, Q^*, \{\langle 1 - \nu(p, q), \nu(p, q) \rangle \mid p \in P, q \in Q\}] \\
 &= \diamond A \oplus_V \diamond B;
 \end{aligned}$$

**Cm2** As it is proved in [?], for each IFS  $X$ :

$$\diamond\diamond X = \diamond X$$

and from this equality it follows:

$$\begin{aligned}\diamond\diamond A &= \diamond[\diamond K^*, \diamond L^*, \diamond\{\langle\mu(k, l), \nu(k, l)\rangle \mid k \in K, l \in L\}] \\ &= [\diamond\diamond K^*, \diamond\diamond^* L, \diamond\{\langle 1 - \nu(k, l), \nu(k, l)\rangle \mid k \in K, l \in L\}] \\ &= [\diamond K^*, \diamond^* L, \{\langle 1 - \nu(k, l), \nu(k, l)\rangle \mid k \in K, l \in L\}] \\ &= \diamond A;\end{aligned}$$

**Cm3**

$$\begin{aligned}A &= [K^*, L^*, \{\langle\mu(k, l), \nu(k, l)\rangle \mid k \in K, l \in L\}] \\ &\subseteq_{v,i} [\diamond K^*, \diamond L^*, \{\langle 1 - \nu(k, l), \nu(k, l)\rangle \mid k \in K, l \in L\}] \\ &= \diamond A.\end{aligned}$$

## Cm4

$$\begin{aligned}
 \diamond E^* &= \diamond [I_1^*, I_2^*, \diamond \{\langle 1, 0 \rangle \mid k \in K, l \in L\}] \\
 &= [I_1^*, I_2^*, \{\langle 1, 0 \rangle \mid k \in K, l \in L\}] \\
 &= E^*.
 \end{aligned}$$

(\*) For each IFS  $X$ :  $\mathcal{C}(\diamond X) = \diamond \mathcal{C}(X)$  and from this equality it follows:

$$\begin{aligned}
 \mathcal{C}(\diamond A)) &= \mathcal{C}([\diamond K^*, \diamond L^*, \{\langle 1 - \nu(k, l), \nu(k, l) \rangle \mid k \in K, l \in L\}]) \\
 &= [\mathcal{C}(\diamond K^*), \mathcal{C}(\diamond L^*), \{\langle \sup_{\langle k, l \rangle \in K \times L} (1 - \nu(k, l)), \\
 &\quad \inf_{\langle k, l \rangle \in K \times L} \nu(k, l) \rangle \mid k \in K, l \in L\}]) \\
 &= [\diamond \mathcal{C}(K^*), \diamond \mathcal{C}(L^*), \{\langle 1 - \inf_{\langle k, l \rangle \in K \times L} (1 - \nu(k, l)), \\
 &\quad \inf_{\langle k, l \rangle \in K \times L} \nu(k, l) \rangle \mid k \in K, l \in L\}]) \\
 &= \diamond \mathcal{C}(A).
 \end{aligned}$$

The remaining 15 assertions are proved in the same manner.

Obviously, if  $\mathcal{I}^* = \mathcal{I}_1^* \cup \mathcal{I}_2^*$ , the above theorem can be modified to the form for the case of IM2DMTS.

# Intuitionistic fuzzy interpretation of multi-person multi-criteria decision making procedures

In a previous research, a very general multi-person multi-criteria decision making procedure is described. We will use it as a basis of the next research in which elements of this procedure will be used.

Let us have  $e$  experts  $X_1, X_2, \dots, X_e$  who must evaluate (on first step) only one object, using  $c$  criteria  $C_1, C_2, \dots, C_c$ .

Let each expert have his/her own (current) reliability score in the form of an IFP  $\langle \delta_i, \varepsilon_i \rangle$  and his/her own (current) number of participations in expert investigations  $\gamma_i$  (these two values correspond to his last expert estimation). Expert's reliability scores can be interpreted, e.g., as

$$\langle \delta_i, \varepsilon_i \rangle = \left\langle \frac{\sum_{j=1}^e \delta_{i,j}}{e}, \frac{\sum_{j=1}^e \varepsilon_{i,j}}{e} \right\rangle,$$

where  $\langle \delta_{i,j}, \varepsilon_{i,j} \rangle$  are elements of the IM

$$T = \begin{array}{c|ccccc} & X_1 & X_2 & \dots & X_e \\ \hline C_1 & & & & & \langle \delta_{i,j}, \varepsilon_{i,j} \rangle \\ C_2 & & & & & (1 \leq i \leq c, \\ \vdots & & & & & \\ C_c & & & & & 1 \leq j \leq e) \end{array}$$

and  $\langle \delta_{i,j}, \varepsilon_{i,j} \rangle$  is the score of the  $i$ -th expert with respect to the  $j$ -th criterion (we assume that  $i$ -th expert's knowledge reliability may differ over different criteria; the case when the expert is equally good a specialist with respect to the different criteria is a special one).

Now, we can construct an EIFIM

	$X_1, \langle \delta_1, \varepsilon_1 \rangle$	$\dots$	$X_j, \langle \delta_j, \varepsilon_j \rangle$	$\dots$	$X_e, \langle \delta_e, \varepsilon_e \rangle$
$C_1, \langle \alpha_1, \beta_1 \rangle$	$\langle \mu_{C_1, X_1}, \nu_{C_1, X_1} \rangle$	$\dots$	$\langle \mu_{C_1, X_j}, \nu_{C_1, X_j} \rangle$	$\dots$	$\langle \mu_{C_1, X_e}, \nu_{C_1, X_e} \rangle$
$\vdots$	$\vdots$	$\dots$	$\vdots$	$\dots$	$\vdots$
$C_i, \langle \alpha_i, \beta_i \rangle$	$\langle \mu_{C_i, X_1}, \nu_{C_i, X_1} \rangle$	$\dots$	$\langle \mu_{C_i, X_j}, \nu_{C_i, X_j} \rangle$	$\dots$	$\langle \mu_{C_i, X_e}, \nu_{C_i, X_e} \rangle$
$\vdots$	$\vdots$	$\dots$	$\vdots$	$\dots$	$\vdots$
$C_c, \langle \alpha_c, \beta_c \rangle$	$\langle \mu_{C_c, X_1}, \nu_{C_c, X_1} \rangle$	$\dots$	$\langle \mu_{C_c, X_j}, \nu_{C_c, X_j} \rangle$	$\dots$	$\langle \mu_{C_c, X_e}, \nu_{C_c, X_e} \rangle$

Having in mind the results from Section 3, we see that object  $\langle \mathcal{P}(C^* \times X^* \times E^*), \mathcal{E}, \zeta, *, \eta \rangle$  is an IF3DMTS, where  $\mathcal{E} \in \{\mathcal{C}, \mathcal{I}\}$  is one of the topological operators,  $* \in \{ \square, \diamond \}$  is one of the modal operators,  $\zeta, \eta \in \{ \vee, \wedge \}$  and

$$C^* = \{ \langle C_i, \alpha_i, \beta_i \rangle \mid 1 \leq i \leq c \},$$

$$X^* = \{ \langle X_j, \delta_j, \varepsilon_j \rangle \mid 1 \leq j \leq e \},$$

$$E^* = \{ \langle \langle C_i, X_j \rangle, \mu_{C_i, X_j}, \nu_{C_i, X_j} \rangle \mid 1 \leq i \leq c, 1 \leq j \leq e \},$$

are IFSs

Now, we can take the next step – the experts will evaluate many objects, e.g.,  $s$  in number –  $S = \{S_1, S_2, \dots, S_s\}\}$ . In this case the EIFIM will be 3-dimensional (see Fig. 2).

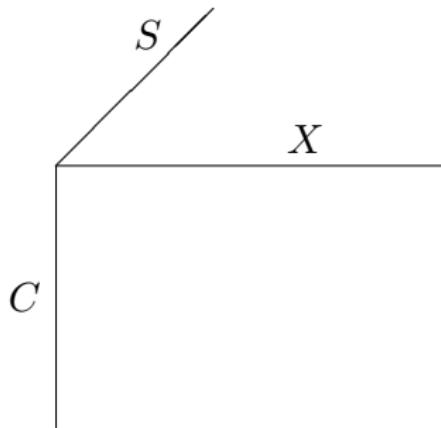


Fig. 2: A 3-dimensional EIFIM

The  $q$ -th level of this parallelepiped, where  $1 \leq q \leq s$  will be related to the experts' evaluations of the object  $S_q$  and will have the form

$S_q$	$X_1, \langle \delta_1, \varepsilon_1 \rangle$	$\dots$	$X_j, \langle \delta_j, \varepsilon_j \rangle$	$\dots$
$C_1, \langle \alpha_1, \beta_1 \rangle$	$\langle \mu_{C_1, X_1, S_q}, \nu_{C_1, X_1, S_q} \rangle$	$\dots$	$\langle \mu_{C_1, X_j, S_q}, \nu_{C_1, X_j, S_q} \rangle$	$\dots$
$\vdots$	$\vdots$	$\dots$	$\vdots$	$\dots$
$C_i, \langle \alpha_i, \beta_i \rangle$	$\langle \mu_{C_i, X_1, S_q}, \nu_{C_i, X_1, S_q} \rangle$	$\dots$	$\langle \mu_{C_i, X_j, S_q}, \nu_{C_i, X_j, S_q} \rangle$	$\dots$
$\vdots$	$\vdots$	$\dots$	$\vdots$	$\dots$
$C_c, \langle \alpha_c, \beta_c \rangle$	$\langle \mu_{C_c, X_1, S_q}, \nu_{C_c, X_1, S_q} \rangle$	$\dots$	$\langle \mu_{C_c, X_j, S_q}, \nu_{C_c, X_j, S_q} \rangle$	$\dots$

In this EIFIM the set

$$S^* = \{\langle S_q, \varphi_q, \psi_q \rangle \mid 1 \leq q \leq s\}$$

is an IFS and the object  $\langle \mathcal{P}(C^* \times X^* \times S^* \times E^*), \mathcal{E}, \zeta, *, \eta \rangle$  is an IF4DMTS.

## Conclusion

We will finish the present paper with short remarks for a future research.

First, we can further complicate our construction by introducing a time components. In a result we will obtain the IF4DMTS

$\mathcal{P}(C^* \times X^* \times T \times E^*)$  in which only the time-scale  $T$  will be not intuitionistic fuzzified.

Second, over the set  $\mathcal{P}(C^* \times X^* \times E^*)$  or  $\mathcal{P}(C^* \times X^* \times S^* \times E^*)$  we can apply the level operators introduced over IFSs, IFIMs and EIFIMs. In this case, we can construct IFMDM\_level\_TS, that will be an extension of the IFMLTS.

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**Thank you for attention!**