Intuitionistic fuzzy interpretation of a classical propositional logic formula

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Introduction

After the first classical and non-classical intuitionistic fuzzy implications, introduced in the end of 1980s, more than 200 different intuitionistic fuzzy implications were introduced. Each one of these implications generates an intuitionistic fuzzy negation and now there are more than 50 different negations.

In a series of papers the properties of these implications and negations are studied.

The following formula has been discussed in the literature:

$$\neg A = (A \to ((A \to A) \wedge \neg(A \to A))). \qquad (*)$$

Obviously, formula (*) is a tautology in the classical propositional logic.
In the present research, we will check which intuitionistic fuzzy implications and negations satisfy (*) when the evaluation of the variable A is an intuitionistic fuzzy pair, i.e., this evaluation has the form of an intuitionistic fuzzy pair:

$$V(A) = \langle a, b \rangle,$$

where $a, b \in [0, 1]$ and $a + b \leq 1$.

We will use two different pairs of operations "implication" and "negation". In the first case, the implications will be different, while the negation will be only the classical intuitionistic fuzzy negation, that is defined by

$$\neg\langle a, b\rangle = \neg_1\langle a, b\rangle = \langle b, a\rangle.$$

In the second case, the implications will be different and the negations will be generated for the respective implication by the formula

$$\neg_{\varphi(i)}\langle a, b\rangle = \langle a, b\rangle \rightarrow_i \langle 0, 1\rangle,$$

where $\varphi(i)$ is the number of the negation generated by the i-th intuitionistic fuzzy implication.

The pair of an intuitionistic fuzzy implication and it specific negation generates intuitionistic fuzzy conjunction as follows:

$$\langle a,b\rangle \wedge_{i,1} \langle c,d\rangle = \neg(\langle a,b\rangle \rightarrow_i \neg\langle c,d\rangle),$$
$$\langle a,b\rangle \wedge_{i,2} \langle c,d\rangle = \neg_{\varphi(i)}(\langle a,b\rangle \rightarrow_i \neg_{\varphi(i)}\langle c,d\rangle),$$
$$\langle a,b\rangle \wedge_{i,3} \langle c,d\rangle = \neg_{\varphi(i)}(\neg_{\varphi(i)}\neg_{\varphi(i)}\langle a,b\rangle \rightarrow_i \neg_{\varphi(i)}\langle c,d\rangle).$$

The third cases are different than the second cases when the intuitionistic fuzzy negation does not satisfy the De Morgan's laws.

Main results

Below, we will formulate and prove three assertions.

Theorem 1. *For* $i = 1, 4, 7, 10, 19, 61, 63, 67, 68, 69, 70, 73, 166, 186, 192$, *formula* (*) *is valid for the intuitionistic fuzzy implication* $\rightarrow_i$, *classical negation* $\neg_1$ *and conjunction* $\wedge_{i,1}$.

Proof. Below, we will prove the validity of the assertion, when $i = 1$. This first implication is an intuitionistic fuzzy modification of Zadeh's implication and it has the form:

$$\langle a, b \rangle \rightarrow_1 \langle c, d \rangle = \langle \max(b, \min(a, c)), \min(a, d) \rangle.$$

This intuitionistic fuzzy implication generates the classical intuitionistic fuzzy negation $\neg_1$ or (as usually, denoted for brevity) $\neg$. From this fact it follows that the three forms of the disjunction and conjunction coincide. They have the forms:

$$\langle a, b\rangle \vee_1 \langle c, d\rangle = \langle \max(a, \min(b, c)), \min(b, d)\rangle,$$

$$\langle a, b\rangle \wedge_1 \langle c, d\rangle = \langle \min(a, c), \max(b, \min(a, d))\rangle.$$

Now, the left hand-side of (*) has the form:

$$V(\neg A) = \langle b, a\rangle,$$

while the right hand-side of (*) has the form:

$$
\begin{aligned}
& V(A \to_1 ((A \to_1 A) \wedge_{1,1} \neg_1 (A \to_1 A))) \\
& = \langle a, b \rangle \to_1 (((\langle a, b \rangle \to_1 \langle a, b \rangle) \wedge_{1,1} \neg_1 (\langle a, b \rangle \to_1 \langle a, b \rangle)) \\
& = \langle a, b \rangle \to_1 (\langle \max(b, \min(a, a)), \min(a, b) \rangle \wedge_{1,1} \\
& \qquad\qquad \neg_1 \langle \max(b, \min(a, a)), \min(a, b) \rangle) \\
& = \langle a, b \rangle \to_1 (\langle \max(a, b), \min(a, b) \rangle \wedge_{1,1} \neg_1 \langle \max(a, b), \min(a, b) \rangle) \\
& = \langle a, b \rangle \to_1 (\langle \max(a, b), \min(a, b) \rangle \wedge_{1,1} \langle \min(a, b), \max(a, b) \rangle) \\
& = \langle a, b \rangle \to_1 \langle \min(\max(a, b), \min(a, b)), \\
& \qquad\qquad \max(\min(a, b), \min(\max(a, b), \max(a, b))) \rangle \\
& = \langle a, b \rangle \to_1 \langle \min(a, b), \max(\min(a, b), \max(a, b)) \rangle \\
& = \langle a, b \rangle \to_1 (\langle \min(a, b), \max(a, b) \rangle \\
& = \langle \max(b, \min(a, \min(a, b))), \min(a, \max(a, b)) \rangle \\
& = \langle \max(b, \min(a, b)), a \rangle \\
& = \langle b, a \rangle,
\end{aligned}
$$

i.e., the left-hand side and the right-hand side of (*) coincide and the equality is valid.

The checks of the remaining equalities are similar.

We can mention that formula (*) will also be valid, if we use the implication $\rightarrow_1$, negation $\neg$ and the most popular form of the conjunction:

$$\langle a, b\rangle \wedge \langle c, d\rangle = \langle \min(a, c), \max(b, d))\rangle.$$

In this case, the check is the following:

$$
\begin{aligned}
& V(A \to_1 ((A \to_1 A) \wedge \neg(A \to_1 A))) \\
& = \langle a, b\rangle \to_1 (((\langle a, b\rangle \to_1 \langle a, b\rangle) \wedge \neg(\langle a, b\rangle \to_1 \langle a, b\rangle)) \\
& = \langle a, b\rangle \to_1 (\langle \max(b, \min(a, a)), \min(a, b)\rangle \\
& \qquad\qquad\qquad \wedge \neg\langle \max(b, \min(a, a)), \min(a, b)\rangle) \\
& = \langle a, b\rangle \to_1 (\langle \max(a, b), \min(a, b)\rangle \wedge \neg\langle \max(a, b), \min(a, b)\rangle) \\
& = \langle a, b\rangle \to_1 (\langle \max(a, b), \min(a, b)\rangle \wedge \langle \min(a, b), \max(a, b)\rangle) \\
& = \langle a, b\rangle \to_1 \langle \min(\max(a, b), \min(a, b)), \max(\min(a, b), \max(a, b))\rangle \\
& = \langle a, b\rangle \to_1 \langle \min(a, b), \max(a, b)\rangle \\
& = \langle \max(b, \min(a, \min(a, b))), \min(a, \max(a, b))\rangle \\
& = \langle \max(b, \min(a, b))), a\rangle \\
& = \langle b, a\rangle.
\end{aligned}
$$

Let us define for each $x \in [0, 1]$:

$$\mathrm{sg}(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x \leq 0 \end{cases},$$

and

$$\overline{\mathrm{sg}}(x) = \begin{cases} 0, & \text{if } x > 0 \\ 1, & \text{if } x \leq 0 \end{cases}.$$

Theorem 2. *For* $i = 1, 2, 3, 4, 7, 8, 10, 11, 14, 15, 16, 18, 19, 20, 22, 23, 24, 26, 27, 30, 31, 32, 33, 34, 37, 39, 40, 41, 42, 43, 44, 45, 47, 48, 49, 52, 55, 56, 57, 58, 59, 60, 61, 62, 63, 65, 67, 68, 69, 70, 73, 74, 76, 77, 79, 81, 83, 84, 87, 88, 89, 90, 92, 93, 96, 97, 99, 100, 101, 102, 104, 105, 107, 108, 109, 114, 166, 167, 168, 170, 171, 172, 173, 174, 175, 176, 177, 178, 179, 180, 181, 182, 183, 184, 185, 186, 187, 188, 189, 190, 191, 192, 194, 195, 198, 199, 200, 201, 202, 203, 205, 206,$ *formula* (*) *is valid for the intuitionistic fuzzy implication* $\rightarrow_i$*, negation* $\neg_{\varphi(i)}$ *and conjunction* $\wedge_{i,2}$.

Proof.

Below, we will prove the validity of the assertion, when $i = 2$. This second implication is an intuitionistic fuzzy modification of Gaines–Rescher's implication and it has the form:

$$\langle a, b\rangle \rightarrow_2 \langle c, d\rangle = \langle \overline{\text{sg}}(a-c), d\,\text{sg}(a-c)\rangle.$$

It generates the intuitionistic fuzzy negation $\neg_2$ that has the form:

$$\neg_2\langle a, b\rangle = \langle \overline{\text{sg}}(a), \text{sg}(a)\rangle.$$

Then, having in mind that for each $x \in [0, 1]$ we can check directly that:

$$\text{sg}(\overline{\text{sg}}(x)) = \overline{\text{sg}}(x)$$

and

$$\overline{\text{sg}}(\overline{\text{sg}}(x)) = \text{sg}(x),$$

for the three forms of the second conjunction we obtain:

$$\begin{aligned}
\langle a,b\rangle \wedge_{2,1} \langle c,d\rangle &= \neg(\langle a,b\rangle \rightarrow_2 \neg\langle c,d\rangle)\\
&= \neg\langle \overline{\mathrm{sg}}(a-c), d\,\mathrm{sg}(a-c)\rangle\\
&= \langle \overline{\mathrm{sg}}(\overline{\mathrm{sg}}(a-c)), \mathrm{sg}(\overline{\mathrm{sg}}(a-c))\rangle\\
&= \langle \mathrm{sg}(a-c), d\,\overline{\mathrm{sg}}(a-c)\rangle
\end{aligned}$$

$$\begin{aligned}
\langle a,b\rangle \wedge_{2,2} \langle c,d\rangle &= \neg_2(\langle a,b\rangle \rightarrow_2 \neg_2\langle c,d\rangle)\\
&= \neg_2(\langle a,b\rangle \rightarrow_2 \langle \overline{\mathrm{sg}}(c), \mathrm{sg}(c)\rangle)\\
&= \neg_2\langle \overline{\mathrm{sg}}(a-\overline{\mathrm{sg}}(c)), \mathrm{sg}(c)\mathrm{sg}(a-\overline{\mathrm{sg}}(c))\rangle\\
&= \langle \overline{\mathrm{sg}}(\overline{\mathrm{sg}}(a-\overline{\mathrm{sg}}(c))), \mathrm{sg}(\overline{\mathrm{sg}}(a-\overline{\mathrm{sg}}(c)))\rangle\\
&= \langle \mathrm{sg}(a-\overline{\mathrm{sg}}(c)), \overline{\mathrm{sg}}(a-\overline{\mathrm{sg}}(c))\rangle.
\end{aligned}$$

$$
\begin{aligned}
\langle a,b\rangle \wedge_{2,3} \langle c,d\rangle &= \neg_2(\neg_2\neg_2\langle a,b\rangle \rightarrow_2 \neg_2\langle c,d\rangle)\\
&= \neg_2(\neg_2\langle \overline{\mathrm{sg}}(a), \mathrm{sg}(a)\rangle \rightarrow_2 \neg_2\langle c,d\rangle)\\
&= \neg_2(\langle \overline{\mathrm{sg}}(\overline{\mathrm{sg}}(a)), \mathrm{sg}(\overline{\mathrm{sg}}(a))\rangle \rightarrow_2 \neg_2\langle c,d\rangle)\\
&= \neg_2(\langle \mathrm{sg}(a), \overline{\mathrm{sg}}(a)\rangle \rightarrow_2 \neg_2\langle \overline{\mathrm{sg}}(c), \mathrm{sg}(c)\rangle)\\
&= \neg_2\langle \overline{\mathrm{sg}}(\mathrm{sg}(a) - \overline{\mathrm{sg}}(c)), \mathrm{sg}(c)\mathrm{sg}(\mathrm{sg}(a) - \overline{\mathrm{sg}}(c))\rangle\\
&= \langle \overline{\mathrm{sg}}(\overline{\mathrm{sg}}(\mathrm{sg}(a) - \overline{\mathrm{sg}}(c))), \mathrm{sg}(\overline{\mathrm{sg}}(\mathrm{sg}(a) - \overline{\mathrm{sg}}(c)))\rangle\\
&= \langle \mathrm{sg}(\mathrm{sg}(a) - \overline{\mathrm{sg}}(c)), \overline{\mathrm{sg}}(\mathrm{sg}(a) - \overline{\mathrm{sg}}(c))\rangle.
\end{aligned}
$$

In the present proof, we need only from the form of conjunction $\wedge_{2,2}$, but we shown the three forms that can be used in the detailed proofs of the three theorems. The same we will do in the proof of Theorem 3. Now, the left-hand side of (*) has the form:

$$V(\neg A) = \langle \overline{\text{sg}}(a), \text{sg}(a) \rangle,$$

while the right-hand side of (*) has the form:

$$
\begin{aligned}
&V(A \to ((A \to A) \wedge \neg(A \to A))) \\
&= \langle a, b\rangle \to (((\langle a, b\rangle \to \langle a, b\rangle) \wedge \neg(\langle a, b\rangle \to \langle a, b\rangle)) \\
&= \langle a, b\rangle \to (\langle \overline{\mathrm{sg}}(a - a), b\,\mathrm{sg}(a - a)\rangle \wedge \neg(\langle \overline{\mathrm{sg}}(a - a), b\,\mathrm{sg}(a - a)\rangle)) \\
&= \langle a, b\rangle \to (\langle \overline{\mathrm{sg}}(0), b\,\mathrm{sg}(0)\rangle \wedge \neg\langle \overline{\mathrm{sg}}(0), b\,\mathrm{sg}(0)\rangle) \\
&= \langle a, b\rangle \to (\langle 1, 0\rangle \wedge \neg\langle 1, 0\rangle) \\
&= \langle a, b\rangle \to (\langle 1, 0\rangle \wedge \langle 0, 1\rangle) \\
&= \langle a, b\rangle \to \langle \mathrm{sg}(1 - \overline{\mathrm{sg}}(0)), \overline{\mathrm{sg}}(1 - \overline{\mathrm{sg}}(0))\rangle \\
&= \langle a, b\rangle \to \langle 0, 1\rangle \\
&= \langle \mathrm{sg}(a - 0), \overline{\mathrm{sg}}(a - 0)\rangle \\
&= \langle \overline{\mathrm{sg}}(a), \mathrm{sg}(a)\rangle.
\end{aligned}
$$

Therefore, both sides of (*) coincide. The checks of the remaining equalities are proved in the same manner. □

We must note that no one of the intuitionistic fuzzy implications ($\rightarrow_9, \rightarrow_{17}, \rightarrow_{21}$) related to the intuitionistic fuzzy negation $\neg_3$ satisfy (*). Indeed, they are defined by

$$\langle a, b\rangle \rightarrow_9 \langle c, d\rangle = \langle b + a^2 c, ab + a^2 d\rangle$$

$$\langle a, b\rangle \rightarrow_{17} \langle c, d\rangle = \langle \max(b, c), \min(ab + a^2, d)\rangle$$

$$\langle a, b\rangle \rightarrow_{21} \langle c, d\rangle = \langle \max(b, c(c + d)), \min(a(a + b), d(c^2 + d + cd))\rangle.$$

and the generated by each one of them negation $\neg_3$ is

$$\neg_3\langle a, b\rangle = \langle b, ab + a^2\rangle$$

and

$$\langle a, b\rangle \wedge_{9,1} \langle c, d\rangle = \neg(\langle a, b\rangle \rightarrow_9 \neg\langle c, d\rangle) = \langle ab + a^2 d, b + a^2 c\rangle.$$

Now, for

$$V(A) = \left\langle \frac{1}{4}, 0 \right\rangle$$

we obtain

$$V(\neg A) = \left\langle 0, \frac{1}{4} \times 0 + \left(\frac{1}{4}\right)^2 \right\rangle = \left\langle 0, \frac{1}{16} \right\rangle,$$

while, for example for $\rightarrow_9$ we have

$$
\begin{aligned}
&V(A \to_9 ((A \to_9 A) \wedge_{9,1} \neg_3(A \to_9 A))) \\
&= \langle a, b\rangle \to_9 (((\langle a, b\rangle \to_9 \langle a, b\rangle) \wedge_{9,1} \neg_3(\langle a, b\rangle \to_9 \langle a, b\rangle)) \\
&= \left\langle \tfrac{1}{4}, 0\right\rangle \to_9 (((\left\langle \tfrac{1}{4}, 0\right\rangle \to_9 \left\langle \tfrac{1}{4}, 0\right\rangle) \wedge_{9,1} \neg_3(\left\langle \tfrac{1}{4}, 0\right\rangle \to_9 \left\langle \tfrac{1}{4}, 0\right\rangle)) \\
&= \left\langle \tfrac{1}{4}, 0\right\rangle \to_9 (\left\langle \tfrac{1}{64}, 0\right\rangle \wedge_{9,1} \neg_3 \left\langle \tfrac{1}{64}, 0\right\rangle) \\
&= \left\langle \tfrac{1}{4}, 0\right\rangle \to_9 (\left\langle \tfrac{1}{64}, 0\right\rangle \wedge_{9,1} \left\langle 0, \tfrac{1}{4096}\right\rangle) \\
&= \left\langle \tfrac{1}{4}, 0\right\rangle \to_9 \left\langle 0, \tfrac{1}{16777216}\right\rangle \\
&= \left\langle 0, \tfrac{1}{268435456}\right\rangle \neq \left\langle 0, \tfrac{1}{16}\right\rangle .
\end{aligned}
$$

Theorem 3. *For* $i = 1, 2, 3, 4, 7, 8, 10, 11, 14, 15, 16, 18, 19, 20, 22, 23, 24, 26, 27, 30, 31, 32, 33, 34, 37, 39, 40, 41, 42, 43, 44, 45, 47, 48, 49, 52, 55, 56, 57, 58, 59, 60, 61, 62, 63, 65, 67, 68, 69, 70, 73, 74, 76, 77, 79, 81, 83, 84, 87, 88, 89, 90, 92, 93, 96, 97, 99, 100, 101, 102, 104, 105, 107, 108, 109, 114, 117, 166, 167, 168, 170, 171, 172, 173, 174, 175, 176, 177, 178, 179, 180, 181, 182, 183, 184, 185, 186, 187, 188, 189, 190, 191, 192, 194, 195, 198, 199, 200, 201, 202, 203, 205, 206$, *formula* (*) *is valid for the intuitionistic fuzzy implication* $\rightarrow_i$, *negation* $\neg_{\varphi(i)}$ *and conjunction* $\wedge_{i,3}$.

Proof. Below, we will prove the validity of the assertion, when $i = 18$. This implication has the form:

$$\langle a, b\rangle \to_{18} \langle c, d\rangle = \langle \max(b, c), \min(1 - b, d)\rangle.$$

It generates the intuitionistic fuzzy negation $\neg_4$ (because $\varphi(18) = 4$) that has the form:

$$\neg_4\langle a, b\rangle = \langle b, 1 - b\rangle.$$

In this case

$$\begin{aligned}\langle a,b\rangle \wedge_{18,1} \langle c,d\rangle &= \neg(\langle a,b\rangle \to_{18} \neg\langle c,d\rangle)\\ &= \neg(\langle a,b\rangle \to_{18} \langle d,1-d\rangle)\\ &= \neg\langle \max(b,d), \min(1-b,1-d)\rangle\\ &= \langle \min(1-b,1-d), 1-\min(1-b,1-d)\rangle\\ &= \langle 1-\max(b,d), \max(b,d)\rangle,\end{aligned}$$

$$\begin{aligned}\langle a,b\rangle \wedge_{18,2} \langle c,d\rangle &= \neg_4(\langle a,b\rangle \to_{18} \neg_4\langle c,d\rangle)\\ &= \neg_4(\langle a,b\rangle \to_{18} \langle d,1-d\rangle)\\ &= \neg_4\langle \max(b,d), \min(1-b,1-d)\rangle\\ &= \neg_4\langle \max(b,d), 1-\max(b,d)\rangle\\ &= \langle 1-\max(b,d), \max(b,d)\rangle,\end{aligned}$$

and

$$
\begin{aligned}
\langle a, b\rangle \wedge_{18,3} \langle c, d\rangle &= \neg_4(\neg_4\neg_4\langle a, b\rangle \to_{18} \neg_4\langle c, d\rangle) \\
&= \neg_4(\neg_4\langle b, 1-b\rangle \to_{18} \neg_4\langle c, d\rangle) \\
&= \neg_4(\langle 1-b, b\rangle \to_{18} \langle d, 1-d\rangle) \\
&= \neg_4\langle \max(b,d), \min(1-b, 1-d)\rangle \\
&= \neg_4\langle \max(b,d), 1-\max(b,d)\rangle \\
&= \langle 1-\max(b,d), \max(b,d)\rangle.
\end{aligned}
$$

Now, the left-hand side of (*) has the form:

$$V(\neg A) = \langle b, 1-b\rangle,$$

while the right-hand side of (*) has the form:

$$
\begin{aligned}
&(A \to_{18} ((A \to_{18} A) \wedge_{18,3} \neg_4 (A \to_{18} A))) \\
&= \langle a, b\rangle \to_{18} (((\langle a, b\rangle \to_{18} \langle a, b\rangle) \wedge_{18,3} \neg_4 (\langle a, b\rangle \to_{18} \langle a, b\rangle)) \\
&= \langle a, b\rangle \to_{18} (\langle \max(a, b), \min(1-b, b)\rangle \\
&\qquad\qquad \wedge_{18,3} \neg_4 (\langle \max(a, b), \min(1-b, b)\rangle)) \\
&= \langle a, b\rangle \to_{18} (\langle \max(a, b), \min(1-b, b)\rangle \\
&\qquad\qquad \wedge_{18,3} \langle \min(1-b, b), 1 - \min(1-b, b)\rangle) \\
&= \langle a, b\rangle \to_{18} (\langle \max(a, b), \min(1-b, b)\rangle \\
&\qquad\qquad \wedge_{18,3} \langle \min(1-b, b), \max(1-b, b)\rangle) \\
&= \langle a, b\rangle \to_{18} \langle 1 - \max(\min(1-b, b), \max(1-b, b)), \\
&\qquad\qquad \max(1 - \min(1-b, b), \max(1-b, b))\rangle \\
&= \langle a, b\rangle \to_{18} \langle 1 - \max(1-b, b), \max(1-b, b)\rangle \\
&= \langle \max(b, \min(1-b, b)), \min(1-b, \max(1-b, b))\rangle \\
&= \langle b, 1-b\rangle.
\end{aligned}
$$

Therefore, both sides of (*) coincide.

The checks of the remaining equalities are proved in the same manner.

□

Conclusion

The introduction of this logical formula establishes an additional criterion for evaluating the correctness of the implications we have defined. Among the set of good implications, those numbered by $5, 9, 13, 17, 28, 71, 110, 112, 125$ do not satisfy the formula under discussion in at least one of the cases.

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