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On a family of billiards-inspired operators over intuitionistic fuzzy sets

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Introduction

Intuitionistic fuzzy sets (IFSs) were introduced in 1983 by K. Atanassov. An early investigation of the geometrical interpretation of the elements of the intuitionistic fuzzy set was made in 1989. Subsequently many operations and operators (modal (or extended modal), topological, level-type operators) have been proposed over IFSs and many properties of theirs have been investigated. Due to the capability of IFSs, unlike ordinary fuzzy sets, to meaningfully represent their elements as points in a planar interpretational triangle, over the years, there has been a solid body of theoretical research on IFSs,

Introduction

- inspired specifically by the unique geometrical interpretation in the orthogonal unit triangle;
- inspired by the geometry of triangle in general;
- in three-dimensional settings;
- in other geometry-inspired settings.

Later the geometrical interpretations of IFSs have been reinvented for the case of interval-valued intuitionistic fuzzy sets, some new operators over IFS have been proposed, and some software implementations of intuitionistic fuzzy sets and their operators have been further developed.

Thus, the operators proposed here, while having purely geometrical and recreational motivation, driven more by a pure intellectual curiosity rather than a specific utilitarian purpose, appear a next step in a long standing tradition of researching how the geometrical visualization can pay back contributions to the theoretical foundations of IFSs.

A family of billiards-inspired operators

we will briefly remind some notions necessary for the exposition of our idea.

Definition (cf. Atanassov (2012))

Let $A \subset X$, where X is a non-empty universal set. Then an intuitionistic fuzzy set is an object of the form

$$A^* = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in X\}, \tag{1}$$

where $\mu_A : X \to [0,1]$, $\nu_A : X \to [0,1]$ are mappings (dependent on the set A), such that

$$\forall x \in X \;\; 0 \leq \mu_A(x) + \nu_A(x) \leq 1. \tag{2}$$

We call $\mu_A(x)$ **degree of membership** of the element x to the set A. Similarly, we call $\nu_A(x)$ **degree of non-membership** of the element x to the set A. The quantity $\pi_A(x) \stackrel{\text{def}}{=} 1 - \mu_A(x) - \nu_A(x)$ is said to be **hesitancy margin** or **degree of indeterminacy** of the element x.

As is customary, below for brevity we shall use A (instead of A^*) as a denotation for the IFS, since there is no risk of misunderstanding. Similarly, when we refer to an IFS B, C, etc., we will understand the IFS defined by (1) whose mappings depend on the crisp set B, C, etc., respectively.

Let us be given an IFSs A and a parameter $\lambda > 0$. Further we will introduce a family of operators depending on an IFS B (defined over the same universe as A) and λ, acting on A. These operators will require the geometrical interpretation of the intuitionistic fuzzy triangle

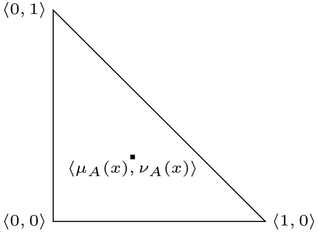

Figure 1: Intuitionistic fuzzy interpretational triangle

For brevity, we will further denote by $r_{AB}(x)$, the vector with tail at $\langle \mu_B(x), \nu_B(x) \rangle$ and head at $\langle \mu_A(x), \nu_A(x) \rangle$. Since both points are inside the interpretational triangle and it is a convex set we easily see that any such vector lies completely inside the interpretational triangle (an example of such vector is shown on the following Figure)

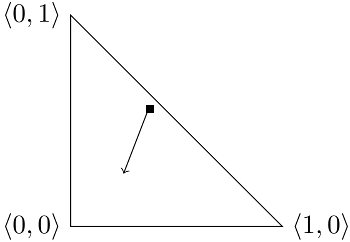

Figure 2: The vector $r_{AB}(x)$ with tail at $\langle 0.375, 0.575 \rangle$ and head at $\langle 0.25, 0.25 \rangle$

Let $|r_{AB}(x)| \neq 0$, then

$$|(1+\lambda)r_{AB}(x)| = (1+\lambda)|r_{AB}(x)| > |r_{AB}(x)|. \tag{3}$$

Thus, the vector $(1+\lambda)r_{AB}(x)$, may have its head outside the interpretational triangle. Since we would like to use the head of this vector as the result of application of our billiards-inspired operator, this is unfortunate. However, the situation is rectifiable by considering the lines passing through the points $\langle 0,0\rangle$ and $\langle 0,1\rangle$; $\langle 0,0\rangle$ and $\langle 1,0\rangle$; $\langle 0,1\rangle$ and $\langle 1,0\rangle$ as reflective surfaces. Therefore, upon reaching any of these surfaces the vector is reflected back inside the triangle.

the special points $\langle 0, 0\rangle$, $\langle 1, 0\rangle$, $\langle 0, 1\rangle$ is that they always reflect a vector by reversing its direction.

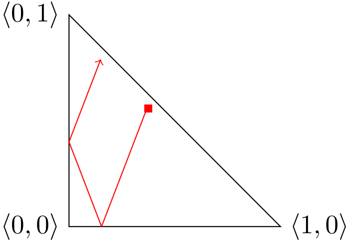

Figure 3: Vector $(1 + \lambda)r_{AB}(x)$ reflected back inside the triangle twice ($\lambda = 3.2$)

Now we are ready to introduce our formal definition.

Definition

Let an IFS A over X be given and a parameter $\lambda > 0$ be fixed. Then for an arbitrary IFS B over X, we define the operator $\mathcal{B}_{B,\lambda}(A)$ as follows:

$$\mathcal{B}_{B,\lambda}(A) = \{\langle x, \mu_{\mathcal{B}_{B,\lambda}}(x), \nu_{\mathcal{B}_{B,\lambda}}(x)\rangle | x \in X\},$$

where $\langle \mu_{\mathcal{B}_{B,\lambda}}(x), \nu_{\mathcal{B}_{B,\lambda}}(x)\rangle =$

$$\begin{cases} \text{the head of } (1+\lambda)r_{AB}(x) \text{ if it lies within the triangle} \\ \text{the head of } s_k, \text{ otherwise} \end{cases} \quad (4)$$

where s_0 is the vector with tail at $\langle \mu_B(x), \nu_B(x)$ and head at the first point of reflection and s_i, $i = 1, 2, \ldots, k$ are the consecutive reflections of $(1+\lambda)r_{AB}(x)$ with

$$\sum_{j=0}^{k} |s_j| = |(1+\lambda)r_{AB}(x)| = (1+\lambda)|r_{AB}(x)|.$$

From the preceding Definition, it is clear that $\mathcal{B}_{B,\lambda}(A)$ is an IFS since from (4) it is evident that the vector's head is in the triangle, i.e. (2) must be fulfilled.
Clearly, all $x \in X$ for which $\mu_A(x) = \mu_B(x)$ and $\nu_A(x) = \nu_B(x)$, act as fixed points for the defined operator. Hence, for any λ, we obviously have:

$$\mathcal{B}_{A,\lambda}(A) = A \tag{5}$$

What is not immediately evident is whether for a fixed choice of λ there is always some an appropriate IFS B (different from A) such that it is fulfilled:

$$\mathcal{B}_{B,\lambda}(A) = A.$$

As a partial resolution we offer the following:

Proposition

For any IFS A and $\lambda = 2$, we can find an IFS B (different from A) such that

$$\mathcal{B}_{B,2}(A) = A.$$

Before we proceed with the proof, we will require the formulation and proof of the following two lemmas.

Lemma

Let $\langle \mu_A(x), \nu_A(x) \rangle$ lie on the boundary of the IFS interpretational triangle. Then we can always choose a point $\langle \mu_B(x), \nu_B(x) \rangle$ (different from $\langle \mu_A(x), \nu_A(x) \rangle$) such that condition (4) is fulfilled.

Proof of the lemma

From the requirements of the Lemma, we have that $\langle \mu_A(x), \nu_A(x) \rangle$ belongs to at least one of the boundary segments of the IFS interpretational triangle. Thus, it will be sufficient to show that for any point $\langle \mu_A(x), \nu_A(x) \rangle$ on a segment with reflective ends, we can find a point $\langle \mu_B(x), \nu_B(x) \rangle$, such that (4) is fulfilled and the head of the last reflection is $\langle \mu_A(x), \nu_A(x) \rangle$. There are three notable cases:

a) $\langle \mu_A(x), \nu_A(x) \rangle$ is an endpoint of the segment.

b) $\langle \mu_A(x), \nu_A(x) \rangle$ is the midpoint of the segment.

c) $\langle \mu_A(x), \nu_A(x) \rangle$ is none of the above.

Case a) Let $\langle \mu_A(x), \nu_A(x) \rangle$ be an endpoint of the segment. Then, choosing $\langle \mu_B(x), \nu_B(x) \rangle$ to be the other endpoint we have $s_0 = R_{AB}(x)$, $s_1 = R_{BA}(x)$, $s_2 = R_{AB}(x)$ (see the Figure below). We can trivially check that condition (4) also holds.
It is easy to see that any number of even reflections $s_1, \ldots, s_{2t}$, $t = 1, 2, 3, \ldots$, will result in the last reflection having head at $\langle \mu_A(x), \nu_A(x) \rangle$, which means that case a) remains true also for $\lambda = 4, 6, 8, \ldots$.

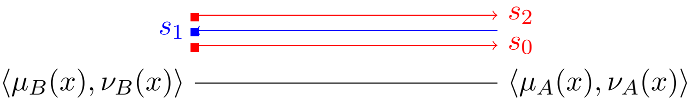

Figure 4: Reflection within a segment: Case a)

Case b) Let $\langle \mu_A(x), \nu_A(x) \rangle$ be the midpoint of the segment. We can again take $\langle \mu_B(x), \nu_B(x) \rangle$ to be any of the endpoints. Then we have $|s_0| + |s_1| = 3|r_{AB}(x)|$ and the head of s_1 is again $\langle \mu_A(x), \nu_A(x) \rangle$ (see the Figure below). Again, the statement remains true for $\lambda = 4, 6, 8, \ldots$

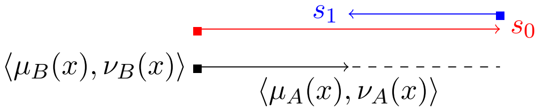

Figure 5: Reflection within a segment: Case b)

Without loss of generality we will assume that $\langle \mu_A(x), \nu_A(x) \rangle$ is closer to the one endpoint whose coordinates we will denote by $\langle h_c, v_c \rangle$. Let us denote the coordinates of the other endpoint by $\langle h_d, v_d \rangle$. Then we must have

$$2\sqrt{(\mu_A(x) - h_c)^2 + (\nu_A(x) - v_c)^2} < \sqrt{(h_d - h_c)^2 + (v_d - v_c)^2}$$

The above ensures that there is a point $\langle \mu_B(x), \nu_B(x) \rangle$ on the segment (see the Figure below), which is different from $\langle h_c, v_c \rangle$, such that $\sqrt{(\mu_A(x) - h_c)^2 + (\nu_A(x) - v_c)^2} = \sqrt{(\mu_B(x) - \mu_A(x))^2 + (\nu_B(x) - \nu_A(x))^2}$.

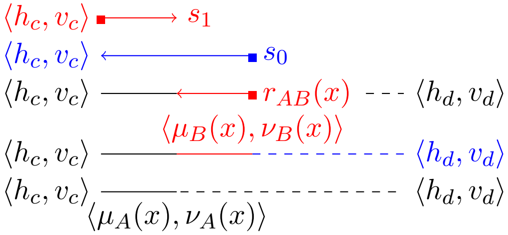

Figure 6: Reflection within a segment: Case c)

Thus, $|s_0| = 2|r_{AB}(x)|$. Hence, $|s_1| = 3|r_{AB}(x)| - |s_0| = |r_{AB}(x)|$. Thus, the head of s_1 must coincide with $\langle \mu_A(x), \nu_A(x) \rangle$. □

Remark. *It is worth noting that in the last case c) of the Lemma , the result in general does not hold for* $\lambda = 4, 6, \ldots$.

Another lemma

Lemma

Let $\langle \mu_A(x), \nu_A(x) \rangle$ *lie in the interior of the IFS interpretational triangle. Then we can always choose a point* $\langle \mu_B(x), \nu_B(x) \rangle$ *(different from* $\langle \mu_A(x), \nu_A(x) \rangle$*) such that condition* (4) *is fulfilled.*

Proof

We will use the idea considered in the proof of the previous Lemma, Case c). For any point $\langle \mu_A(x), \nu_A(x) \rangle$ on or under the line $\nu(x) = \frac{1-\mu(x)}{2}$, the region marked with green in the following Figure, we can find a point $\langle \mu_B(x), \nu_B(x) \rangle$, inside the interpretational triangle such that $\mu_B(x) = \mu_A(x)$ and $\nu_B(x) = 2\nu_A(x)$.

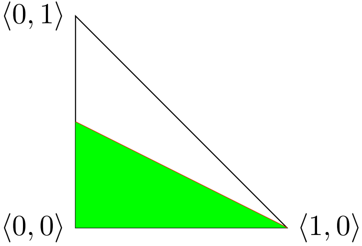

Figure 7: Points that will be reflected onto themselves by the horizontal segment $\langle 0, 0 \rangle$–$\langle 1, 0 \rangle$

Proof (Cont.)

Similarly, for any point $\langle \mu_A(x), \nu_A(x) \rangle$ on or under the line $\nu(x) = 1 - 2\mu(x)$, the region marked with green in the following Figure, we can find a point $\langle \mu_B(x), \nu_B(x) \rangle$, inside the interpretational such that $\mu_B(x) = 2\mu_A(x)$ and $\nu_B(x) = \nu_A(x)$.

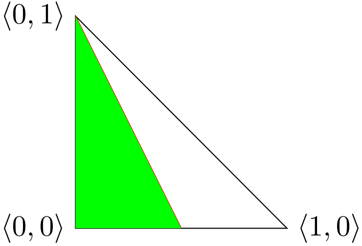

Figure 8: Points that will be reflected onto themselves by the vertical segment $\langle 0, 0 \rangle$–$\langle 0, 1 \rangle$

Proof (Cont.)

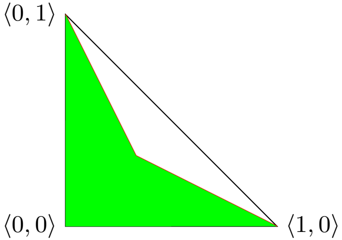

Figure 9: Points that will be reflected onto themselves by one of the legs of the intepretational triangle

Proof (Cont.)

These choices ensure that $3|r_{AB}(x)| = |s_0| + |s_1|$ and the head of s_1 will be $\langle \mu_A(x), \nu_A(x) \rangle$.

Due to the symmetry, without loss of generality we may assume that we have $\mu_A(x) \geq \nu_A(x)$. Therefore, it remains only to show that for any such point $\langle \mu_A(x), \nu_A(x) \rangle$ within the triangle with vertices $\langle \frac{1}{2}, \frac{1}{2} \rangle$, $\langle \frac{1}{3}, \frac{1}{3} \rangle$, $\langle 1, 0 \rangle$, (marked in green in the Figure on the next slide, there exists $\langle \mu_{B'}(x), 0 \rangle$ in the interpretational triangle such that we have:

$$\begin{cases} \mu_{B'}(x) = \mu_A(x) - \nu_A(x) \\ \sqrt{(\mu_{B'}(x) - \mu_A(x))^2 + \nu_A(x)^2} = \sqrt{2}\nu_A(x) \geq \frac{\sqrt{2}}{2}(1 - \mu_A(x) - \nu_A(x)) \end{cases} \tag{6}$$

Proof (Cont.)

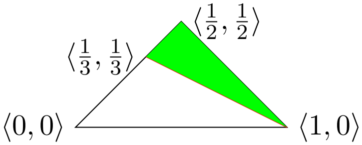

Figure 10: Points that are not reflected on themselves by the legs of the interpretational triangle for which $\mu_A(x) \geq \nu_A(x)$.

Proof (Cont.)

The conditions in (6) ensure that the distance between $\langle \mu_{B'}(x), 0 \rangle$ and $\langle \mu_A(x), \nu_A(x) \rangle$ is greater than or equal to the distance from $\langle \mu_A(x), \nu_A(x) \rangle$ to the segment with endpoints $\langle \frac{1}{2}, \frac{1}{2} \rangle$ and $\langle 1, 0 \rangle$, and ensure that the perpendicular from $\langle \mu_{B'}(x), 0 \rangle$ to that very segment has the same endpoint. Condition (6) can be rewritten as:

$$\begin{cases} \mu_{B'}(x) = \mu_A(x) - \nu_A(x) \\ \nu_A(x) \geq \frac{1}{3}(1 - \mu_A(x)) \end{cases} \tag{7}$$

The first condition is obviously valid since $\mu_A(x) - \nu_A(x) \geq 0$.

Proof (Cont.)

For the second condition, we note that the green region in the above Figure is given by the following inequalities:

$$\begin{cases} \quad \frac{1}{3} \leq \mu(x) \leq 1 \\ \nu(x) \leq 1 - \mu(x) \\ \mu(x) \geq \nu(x) \\ \nu(x) \geq \frac{1}{2}(1 - \mu(x)) \end{cases} \tag{8}$$

In view of the last condition in (8), we must have

$$\nu_A(x) \geq \frac{1}{2}(1 - \mu_A(x)) \geq \frac{1}{3}(1 - \mu_A(x))$$

Thus, we can always find a non-negative number $\mu_{B'}(x)$ such that the distance from it to $\langle \mu_A(x), \nu_A(x) \rangle$ is greater than or equal to the distance from $\langle \mu_A(x), \nu_A(x) \rangle$ to the segment.
Since we want to find one that is exactly at the same distance that $\langle \mu_A(x), \nu_A(x) \rangle$ is from the segment with vertices $\langle \frac{1}{2}, \frac{1}{2} \rangle$ and $\langle 1, 0 \rangle$, which length is $\frac{\sqrt{2}}{2}(1 - \mu_A(x) - \nu_A(x))$, we will find a point $\langle \mu_B(x), \nu_B(x) \rangle$ on the segment with vertices $\langle \mu_{B'}(x), 0 \rangle$ and $\langle \mu_A(x), \nu_A(x) \rangle$, which is at the same distance from $\langle \mu_A(x), \nu_A(x) \rangle$. Each point of the segment can be represented by:

$$\begin{cases} \mu_B(x) = t\mu_{B'}(x) + (1-t)\mu_A(x) = \mu_A(x) - t\nu_A(x) \\ \nu_B(x) = (1-t)\nu_A(x) \end{cases} \tag{9}$$

for some $t \in (0,1)$. In order to find t, we return to the desired distance, hence:

$$\sqrt{(\mu_B(x)-\mu_A(x))^2+(\nu_B(x)-\nu_A(x))^2}=\frac{\sqrt{2}}{2}(1-\mu_A(x)-\nu_A(x))$$

After simplification, we obtain:

$$t=\frac{1-\mu_A(x)-\nu_A(x)}{2\nu_A(x)}.$$

And thus, $\langle \mu_B(x), \nu_B(x)\rangle =$

$$\left\langle \mu_A(x)-\frac{1-\mu_A(x)-\nu_A(x)}{2}, \nu_A(x)-\frac{1-\mu_A(x)-\nu_A(x)}{2}\right\rangle.$$

Thus, for the $r_{AB}(x)$, we will have $3|r_{AB}(x)|=|s_0|+|s_1|$, and as in Case c) of the first Lemma the head of s_1 coincides with $\langle \mu_A(x), \nu_A(x)\rangle$.

Proof of Proposition 1.

From the Lemmas, it follows that for all $x \in X$, we can always find at least a single point $\langle \mu_B(x), \nu_B(x) \rangle \neq \langle \mu_A(x), \nu_A(x) \rangle$, such that the final reflection coincides with $\langle \mu_A(x), \nu_A(x) \rangle$, hence, we can construct a suitable IFS B, for which

$$\mathcal{B}_{B,2}(A) = A.$$

□

We end with the following **Open problem.** *For any given $\lambda > 2$, is it always possible to construct an IFS B with $\langle \mu_A(x), \nu_A(x) \rangle \neq \langle \mu_B(x), \nu_B(x) \rangle$ such that $\mathcal{B}_{B,\lambda}(A) = A$?*

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Thank You for Your Attention!