Intuitionistic fuzzy temporal operators and temporal topological structures

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Introduction

Some years after introducing of the Intuitionistic Fuzzy Sets (IFSs), the Temporal IFSs (TIFSs) were also introduced and over them, the Intuitionistic Fuzzy Temporal Operators (IFTS) were introduced. After introducing the concept of the Modal Topological Structure (MTS), these operators also found place in the MTSs structures. In the present reseach, we will continue the research over the TIFSs, IFTOs and the MTS based over them.

Let a (crisp) set E be fixed and let $A \subset E$ be a fixed set. An IFS A^* in E is an object of the following form

$$A^* = \{\langle x, \mu_A(x), \nu_A(x) \rangle \mid x \in E\},$$

where functions $\mu_A : E \to [0,1]$ and $\nu_A : E \to [0,1]$ define the *degree of membership* and the *degree of non-membership* of the element $x \in E$ to the set A, respectively. For every two IFSs A and B the following relations, operations and operators will be used

$A \subseteq B$ if and only if $(\forall x \in E)(\mu_A(x) \leq \mu_B(x) \;\&\; \nu_A(x) \geq \nu_B(x))$;

$A \supseteq B$ if and only if $B \subseteq A$;

$A = B$ if and only if $(\forall x \in E)(\mu_A(x) = \mu_B(x) \;\&\; \nu_A(x) = \nu_B(x))$;

$A \cap B = \{\langle x, \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x))\rangle \mid x \in E\}$;

$A \cup B = \{\langle x, \max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x))\rangle \mid x \in E\}$;

$\Box A = \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle | x \in E\}$;

$\Diamond A = \{\langle x, 1 - \nu_A(x), \nu_A(x)\rangle \mid x \in E\}$;

$\mathcal{C}(A) = \{\langle x, \sup_{y \in E} \mu_A(y), \inf_{y \in E} \nu_A(y)\rangle \mid x \in E\}$;

$\mathcal{I}(A) = \{\langle x, \inf_{y \in E} \mu_A(y), \sup_{y \in E} \nu_A(y)\rangle \mid x \in E\}$.

The *Temporal IFS (TIFS)* has the following form:

$$A(T) = \{\langle\langle x, t\rangle\mu_A(x,t), \nu_A(x,t)\rangle | \langle x, t\rangle \in E \times T\},$$

where $A \subset E$ is a fixed set, $\mu_A(x,t) + \nu_A(x,t) \leq 1$ for every $\langle x, t\rangle \in E \times T$, $\mu_A(x,t)$ and $\nu_A(x,t)$ are the degrees of membership and non-membership, respectively, of the element $x \in E$ at the time-moment $t \in T$, where T is a given timescale.
All above operations, relations and operators keep their sense over TIFSs. The two intuitionistic fuzzy temporal operators are:

$$\mathcal{C}^*(A(T)) = \{\langle\langle x, t\rangle, \sup_{u\in T} \mu_{A(T)}(x,u), \inf_{u\in T} \nu_{A(T)}(x,u)\rangle \,|\, \langle x, t\rangle \in E\times T\},$$

$$\mathcal{I}^*(A(T)) = \{\langle\langle x, t\rangle, \inf_{u\in T} \mu_{A(T)}(x,u), \sup_{u\in T} \nu_{A(T)}(x,u)\rangle \,|\, \langle x, t\rangle \in E\times T\}.$$

Intuitionistic fuzzy topological structures containing intuitionistic fuzzy temporal operators

Here, following, yet modifying and extending the definition from the previous research, we will define the concept of a Temporal Topological Structure (TTS).
Let us have a fixed set X and let everywhere below

$$\mathcal{P}(X) = \{Y | Y \subseteq X\}.$$

Let T be a fixed temporal scale.
Let O be the minimal element of $\mathcal{P}(X)$. Obviously, X is its maximal element.

Let us have two operations $\Delta, \nabla : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathcal{P}(X)$ defined in such a way that for every two sets $A, B \in \mathcal{P}(X)$:

$$A \nabla B = \neg(\neg A \Delta \neg B),$$

$$A \Delta B = \neg(\neg A \nabla \neg B),$$

where $\neg P$ is the negation of set P.
Let the operation Δ generate the topological operator $\mathcal{O}$ and the operation ∇ generate the topological operator $\mathcal{Q}$.

Now, we will say that the operator $\mathcal{O}$ is a closure (cl)-topological operator if for each $A, B \in \mathcal{P}$:

C1 $\mathcal{O}(A \Delta B) = \mathcal{O}(A) \Delta \mathcal{O}(B)$,

C2 $A \subseteq \mathcal{O}(A)$,

C3 $\mathcal{O}(\mathcal{O}(A)) = \mathcal{O}(A)$,

C4 $\mathcal{O}(O) = O$,

and that the operator $\mathcal{Q}$ is an interior (in)-topological operator if for each $A, B \in \mathcal{P}$:

I1 $\mathcal{Q}(A \nabla B) = \mathcal{Q}(A) \nabla \mathcal{Q}(B)$,

I2 $\mathcal{Q}(A) \subseteq A$,

I3 $\mathcal{Q}(\mathcal{Q}(A)) = \mathcal{Q}(A)$,

I4 $\mathcal{Q}(X) = X$.

We will assume that for every set $A \in \mathcal{P}(X)$:

$$\mathcal{O}(A) = \neg(\mathcal{Q}(\neg A)),$$

$$\mathcal{Q}(A) = \neg(\mathcal{O}(\neg A)).$$

For example, in the intuitionistic fuzzy case, if operation Δ is the operation "union" ($\cup$), then operator $\mathcal{O}$ will be the cl-topological operator $\mathcal{C}$, and if operation ∇ is the operation "intersection" ($\cap$), then operator $\mathcal{Q}$ will be in-topological operator $\mathcal{I}$. Having in mind that the intuitionistic fuzzy topological operators (IFTOs) $\mathcal{C}^*$ and $\mathcal{I}^*$ satisfy conditions C1–C4 and I1–I4, respectively, we can see that operator $\mathcal{C}^*$ is from a cl-type and operation $\mathcal{I}^*$ is from an in-type.

Now, following the definition of a topological structure, we say that the object $\langle \mathcal{P}(X), \mathcal{E}, \zeta, *, \eta \rangle$ is a τ-Temporal φ-Topological Structure (τ-Tφ-TS) over the set X and the temporal scale T, where $\mathcal{E} \in \{\mathcal{O}, \mathcal{Q}\}$ is a topological operator from φ-type generated by operation $\zeta \in \{\Delta, \nabla\}$ and $* \in \{\circ, \bullet\}$ is a temporal operator from τ-type related to operation $\eta \in \{\Delta, \nabla\}$, where $\varphi, \tau \in \{cl, in\}$. Therefore, each one of the both operators (the topological and the temporal) must satisfy the respective C- or the respective I-conditions.

In addition, the topological and the temporal operators must satisfy the following additional condition (*) for each $A \in \mathcal{P}(X)$:

$$*\mathcal{E}(A) = \mathcal{E}(*A) \qquad (*)$$

that it will be used below.

In the present paper, we illustrate the TTS with examples of the area of intuitionistic fuzziness, constructing different Intuitionistic Fuzzy TTSs (IFTTSs).

We must mention that the four theorems, formulated and proved in a previous research are weak about the present definition for a TTS. We will discuss them below.

Theorem

For each universe E and each time-scale T, $\langle \mathcal{P}(E^*(T)), \mathcal{C}, \cup, \mathcal{C}^*, \cup \rangle$ is an IFcl-Tcl-TS.

Proof. Let the IFSs $A(T), B(T) \in \mathcal{P}(E^*(T))$ be given. We will sequentially prove the validity of the nine conditions: C1–C4 for the topological operator $\mathcal{C}$, C1–C4 for the temporal operator $\mathcal{C}^*$, and the condition (*) for the temporal and topological operators.

C1.

$$
\begin{aligned}
&\mathcal{C}(A(T) \cup B(T)) \\
&= \mathcal{C}(\{\langle\langle x,t\rangle, \mu_{A(T)}(x,t), \nu_{A(T)}(x,t)\rangle | \langle x,t\rangle \in E \times T\} \\
&\qquad \cup \{\langle\langle x,t\rangle, \mu_{B(T)}(x,t), \nu_{B(T)}(x,t)\rangle | \langle x,t\rangle \in E \times T\}) \\
&= \mathcal{C}(\{\langle\langle x,t\rangle, \max(\mu_{A(T)}(x,t), \mu_{B(T)}(x,t)), \\
&\qquad\quad \min(\nu_{A(T)}(x,t), \nu_{B(T)}(x,t))\rangle | \langle x,t\rangle \in E \times T\}) \\
&= \{\langle\langle x,t\rangle, \sup_{y\in E} \max(\mu_{A(T)}(y,t), \mu_{B(T)}(y,t)), \\
&\qquad\quad \inf_{y\in E} \min(\nu_{A(T)}(x,t), \nu_{B(T)}(x,t))\rangle | \langle x,t\rangle \in E \times T\}) \\
&= \{\langle\langle x,t\rangle, \max(\sup_{y\in E} \mu_{A(T)}(y,t), \sup_{y\in E} \mu_{B(T)}(y,t)), \\
&\qquad\quad \min(\inf_{y\in E} \nu_{A(T)}(y,t), \inf_{y\in E} \nu_{B(T)}(y,t))\rangle | \langle x,t\rangle \in E \times T\} \\
&= \mathcal{C}(A(T)) \cup \mathcal{C}(B(T));
\end{aligned}
$$

C2.

$$\begin{aligned} A(T) &= \{\langle\langle x,t\rangle, \mu_{A(T)}(x,t), \nu_{A(T)}(x,t)\rangle | \langle x,t\rangle \in E\times T\} \\ &\subseteq \{\langle\langle x,t\rangle, \sup_{y\in E}\mu_{A(T)}(y,t), \inf_{y\in E}\nu_{A(T)}(y,t)\rangle | \langle x,t\rangle \in E\times T\} \\ &= \mathcal{C}(A(T)); \end{aligned}$$

C3. Having in mind that for each fixed $t \in T$: $\sup_{y\in E}\mu_{A(T)}(y,t)$ and $\inf_{y\in E}\nu_{A(T)}(y,t)$ are constants, we obtain that:

$$\begin{aligned} \mathcal{C}(\mathcal{C}(A(T))) &= \mathcal{C}(\{\langle\langle x,t\rangle, \sup_{y\in E}\mu_{A(T)}(y,t), \inf_{y\in E}\nu_{A(T)}(y,t)\rangle | \langle x,t\rangle \in E\times T\}) \\ &= \{\langle\langle x,t\rangle, \sup_{z\in E}\sup_{y\in E}\mu_{A(T)}(y,t), \inf_{z\in E}\inf_{y\in E}\nu_{A(T)}(y,t)\rangle | \langle x,t\rangle \in E\times T\} \\ &= \{\langle\langle x,t\rangle, \sup_{y\in E}\mu_{A(T)}(y,t), \inf_{y\in E}\nu_{A(T)}(y,t)\rangle | \langle x,t\rangle \in E\times T\} \\ &= \mathcal{C}(A(T)). \end{aligned}$$

C4.

$$\begin{aligned}\mathcal{C}(O^*(T)) &= \mathcal{C}(\{\langle\langle x,t\rangle,0,1\rangle|\langle x,t\rangle \in E\times T\}) \\ &= \{\langle\langle x,t\rangle,\sup_{y\in E} 0,\inf_{y\in E} 1\rangle|\langle x,t\rangle \in E\times T\} \\ &= \{\langle x,0,1\rangle|\langle x,t\rangle \in E\times T\} \\ &= O^*(T);\end{aligned}$$

C1.

$$
\begin{aligned}
&\mathcal{C}^*(A(T) \cup B(T)) \\
&= \mathcal{C}^*(\{\langle\langle x,t\rangle, \max(\mu_{A(T)}(x,t), \mu_{B(T)}(x,t)), \\
&\qquad \min(\nu_{A(T)}(x,t), \nu_{B(T)}(x,t))|\rangle \langle x,t\rangle \in E \times T\}) \\
&= \{\langle\langle x,t\rangle, \sup_{u\in T} \max(\mu_{A(T)}(x,u), \mu_{B(T)}(x,u))), \\
&\qquad \inf_{u\in T} \min(\nu_{A(T)}(x,u), \nu_{B(T)}(x,u))\rangle | \langle x,t\rangle \in E \times T\} \\
&= \{\langle\langle x,t\rangle, \max(\sup_{u\in T} \mu_{A(T)}(x,u), \sup_{u\in T} \mu_{B(T)}(x,u)), \\
&\qquad\qquad \min(\inf_{u\in T} \nu_{A(T)}(x,u), \inf_{u\in T} \nu_{B(T)}(x,u)\rangle | \langle x,t\rangle \in E \times T\} \\
&= \{\langle\langle x,t\rangle, \sup_{u\in T} \mu_{A(T)}(x,u), \inf_{u\in T} \nu_{A(T)}(x,u)\rangle | \langle x,t\rangle \in E \times T\} \\
&\qquad\qquad \cup \{\langle\langle x,t\rangle, \sup_{u\in T} \mu_{B(T)}(x,u), \inf_{u\in T} \nu_{B(T)}(x,u)\rangle | \langle x,t\rangle \in E \times T\} \\
&= \mathcal{C}^*(A(T)) \cup \mathcal{C}^*(B(T)).
\end{aligned}
$$

C2.

$$\begin{aligned} A(T) &= \{\langle\langle x,t\rangle, \mu_{A(T)}(x,t), \nu_{A(T)}(x,t)\rangle | \langle x,t\rangle \in E \times T\} \\ &\subseteq \{\langle\langle x,t\rangle, \sup_{u\in T} \mu_{A(T)}(x,u), \inf_{u\in T} \nu_{A(T)}(x,u)\rangle | \langle x,t\rangle \in E \times T\} \\ &= \mathcal{C}^*(A(T)); \end{aligned}$$

C3. Having in mind that for each fixed $x \in T$: $\sup_{u\in T} \mu_{A(T)}(x,u)$ and $\inf_{u\in T} \nu_{A(T)}(x,u)$ are constants, we obtain that:

$$\begin{aligned} &\mathcal{C}^*(\mathcal{C}^*(T)) \\ &= \mathcal{C}^*(\{\langle\langle x,t\rangle, \sup_{u\in T} \mu_{A(T)}(x,u), \inf_{u\in T} \nu_{A(T)}(x,u)\rangle | \langle x,t\rangle \in E \times T\} \\ &= \{\langle\langle x,t\rangle, \sup_{v\in T} \sup_{u\in T} \mu_{A(T)}(x,u), \inf_{v\in T} \inf_{u\in T} \nu_{A(T)}(x,u)\rangle | \langle x,t\rangle \in E \times T\} \\ &= \{\langle\langle x,t\rangle, \sup_{u\in T} \mu_{A(T)}(x,u), \inf_{u\in T} \nu_{A(T)}(x,u)\rangle | \langle x,t\rangle \in E \times T\} \\ &= \mathcal{C}^*(A(T)); \end{aligned}$$

C4.

$$\begin{aligned}\mathcal{C}^*(O^*(T)) &= \mathcal{C}^*(\{\langle\langle x,t\rangle,0,1\rangle\}|\langle x,t\rangle\in E\times T\})\\ &= (\{\langle\langle x,t\rangle,0,1\rangle\}|\langle x,t\rangle\in E\times T\})\\ &= O^*(T)\end{aligned}$$

(*)

$$\begin{aligned}&\mathcal{C}^*(\mathcal{C}(A(T)))\\ &= \mathcal{C}^*(\{\langle\langle x,t\rangle,\sup_{y\in E}\mu_{A(T)}(y,t),\inf_{y\in E}\nu_{A(T)}(y,t)\rangle|\langle x,t\rangle\in E\times T\})\\ &= \{\langle\langle x,t\rangle,\sup_{y\in E}\sup_{u\in T}\mu_{A(T)}(y,u),\inf_{y\in E}\inf_{u\in T}\nu_{A(T)}(y,u)\rangle|\langle x,t\rangle\in E\times T\}\\ &= \mathcal{C}(\{\langle\langle x,t\rangle,\sup_{u\in T}\mu_{A(T)}(x,u),\inf_{u\in T}\nu_{A(T)}(x,u)\rangle|\langle x,t\rangle\in E\times T\})\\ &= \mathcal{C}(\mathcal{C}^*(A(T))).\end{aligned}$$

Theorem

For each universe E and each time-scale T, $\langle \mathcal{P}(E^*(T)), \mathcal{I}, \cap, \mathcal{I}^*, \cap \rangle$ is an IFin-Tin-TS.

Proof. Let the IFSs $A(T), B(T) \in \mathcal{P}(E^*(T))$ be given. We will sequentially prove the validity of the nine conditions: I1–I4 for the topological operator $\mathcal{I}$, I1–I4 for the temporal operator $\mathcal{I}^*$, and the condition (*) for the temporal and topological operators.

I1.

$$
\begin{aligned}
&\mathcal{I}(A(T)\cap B(T))\\
&=\mathcal{I}(\{\langle\langle x,t\rangle,\mu_{A(T)}(x,t),\nu_{A(T)}(x,t)\rangle|\langle x,t\rangle\in E\times T\}\\
&\qquad\cap\{\langle\langle x,t\rangle,\mu_{B(T)}(x,t),\nu_{B(T)}(x,t)\rangle|\langle x,t\rangle\in E\times T\})\\
&=\mathcal{I}(\{\langle\langle x,t\rangle,\min(\mu_{A(T)}(x,t),\mu_{B(T)}(x,t)),\\
&\qquad\qquad\max(\nu_{A(T)}(x,t),\nu_{B(T)}(x,t))\rangle|\langle x,t\rangle\in E\times T\})\\
&=\{\langle\langle x,t\rangle,\inf_{y\in E}\min(\mu_{A(T)}(x,t),\mu_{B(T)}(x,t)),\\
&\qquad\qquad\sup_{y\in E}\max(\nu_{A(T)}(x,t),\nu_{B(T)}(x,t))\rangle|\langle x,t\rangle\in E\times T\})\\
&=\{\langle\langle x,t\rangle,\min(\inf_{y\in E}\mu_{A(T)}(y,t),\inf_{y\in E}\mu_{B(T)}(y,t)),\\
&\qquad\qquad\max(\sup_{y\in E}\nu_{A(T)}(y,t),\sup_{y\in E}\nu_{B(T)}(y,t))\rangle|\langle x,t\rangle\in E\times T\}\\
&=\mathcal{C}(A(T))\cap\mathcal{C}(B(T));
\end{aligned}
$$

I2.

$$
\begin{aligned}
\mathcal{I}(A(T))) &= \{\langle\langle x,t\rangle, \inf_{y\in E}\mu_{A(T)}(y,t), \sup_{y\in E}\nu_{A(T)}(y,t)\rangle | \langle x,t\rangle \in E\times T\}\\
&\subseteq \{\langle\langle x,t\rangle, \mu_{A(T)}(x,t), \nu_{A(T)}(x,t)\rangle | \langle x,t\rangle \in E\times T\}\\
&= A(T);
\end{aligned}
$$

I3.

$$
\begin{aligned}
&\mathcal{I}(\mathcal{I}(A(T)))\\
&= \mathcal{I}(\{\langle\langle x,t\rangle, \inf_{y\in E}\mu_{A(T)}(y,t), \sup_{y\in E}\nu_{A(T)}(y,t)\rangle | \langle x,t\rangle \in E\times T\})\\
&= \{\langle\langle x,t\rangle, \inf_{z\in E}\inf_{y\in E}\mu_{A(T)}(y,t), \sup_{z\in E}\sup_{y\in E}\nu_{A(T)}(y,t)\rangle | \langle x,t\rangle \in E\times T\}\\
&= \{\langle\langle x,t\rangle, \inf_{y\in E}\mu_{A(T)}(y,t), \sup_{y\in E}\nu_{A(T)}(y,t)\rangle | \langle x,t\rangle \in E\times T\}\\
&= \mathcal{I}(A(T)).
\end{aligned}
$$

I4.

$$
\begin{aligned}
\mathcal{I}(E^*(T)) &= \mathcal{C}(\{\langle\langle x,t\rangle,1,0\rangle|\langle x,t\rangle \in E\times T\})\\
&= \{\langle\langle x,t\rangle, \sup_{y\in E} 1, \inf_{y\in E} 0\rangle|\langle x,t\rangle \in E\times T\}\\
&= \{\langle x,1,0\rangle|\langle x,t\rangle \in E\times T\}\\
&= O^*(T);
\end{aligned}
$$

I1.

$$\begin{aligned}
&\mathcal{I}^*(A(T)\cap B(T))\\
&=\mathcal{I}^*(\{\langle\langle x,t\rangle,\min(\mu_{A(T)}(x,t),\mu_{B(T)}(x,t)),\\
&\qquad \max(\nu_{A(T)}(x,t),\nu_{B(T)}(x,t))|\rangle\langle x,t\rangle\in E\times T\})\\
&=\{\langle\langle x,t\rangle,\inf_{u\in T}\min(\mu_{A(T)}(x,u),\mu_{B(T)}(x,u))),\\
&\qquad\qquad \sup_{u\in T}\max(\nu_{A(T)}(x,u),\nu_{B(T)}(x,u))\rangle|\langle x,t\rangle\in E\times T\}\\
&=\{\langle\langle x,t\rangle,\min(\inf_{u\in T}\mu_{A(T)}(x,u),\inf_{u\in T}\mu_{B(T)}(x,u)),\\
&\qquad\qquad \max(\sup_{u\in T}\nu_{A(T)}(x,u),\sup_{u\in T}\nu_{B(T)}(x,u)\rangle|\langle x,t\rangle\in E\times T\}\\
&=\{\langle\langle x,t\rangle,\inf_{u\in T}\mu_{A(T)}(x,u),\sup_{u\in T}\nu_{A(T)}(x,u)\rangle|\langle x,t\rangle\in E\times T\}\\
&\qquad\qquad \cap\{\langle\langle x,t\rangle,\inf_{u\in T}\mu_{B(T)}(x,u),\sup_{u\in T}\nu_{B(T)}(x,u)\rangle|\langle x,t\rangle\in E\times T\}\\
&=\mathcal{I}^*(A(T))\cap\mathcal{I}^*(B(T)).
\end{aligned}$$

I2.

$$\begin{aligned}\mathcal{I}^*(A(T))) &= \{\langle\langle x,t\rangle, \inf_{u\in Y}\mu_{A(T)}(x,u), \sup_{u\in T}\nu_{A(T)}(x,u)\rangle | \langle x,t\rangle \in E\times T\}\\ &\subseteq \{\langle\langle x,t\rangle, \mu_{A(T)}(x,t), \nu_{A(T)}(x,t)\rangle | \langle x,t\rangle \in E\times T\}\\ &= A(T);\end{aligned}$$

I3.

$$\begin{aligned}&\mathcal{I}^*(\mathcal{I}^*(T))\\ &= \mathcal{I}^*(\{\langle\langle x,t\rangle, \inf_{u\in T}\mu_{A(T)}(x,u), \sup_{u\in T}\nu_{A(T)}(x,u)\rangle | \langle x,t\rangle \in E\times T\}\\ &= \{\langle\langle x,t\rangle, \inf_{v\in T}\inf_{u\in T}\mu_{A(T)}(x,u), \sup_{v\in T}\sup_{u\in T}\nu_{A(T)}(x,u)\rangle | \langle x,t\rangle \in E\times T\}\\ &= \{\langle\langle x,t\rangle, \inf_{u\in T}\mu_{A(T)}(x,u), \sup_{u\in T}\nu_{A(T)}(x,u)\rangle | \langle x,t\rangle \in E\times T\}\\ &= \mathcal{C}^*(A(T));\end{aligned}$$

I4.

$$\begin{aligned}\mathcal{I}^*(E^*(T)) &= \mathcal{I}^*(\{\langle\langle x,t\rangle,1,0\rangle\}|\langle x,t\rangle \in E\times T\})\\ &= (\{\langle\langle x,t\rangle,1,0\rangle\}|\langle x,t\rangle \in E\times T\})\\ &= E^*(T)\end{aligned}$$

(*)

$$\begin{aligned}&\mathcal{I}^*(\mathcal{I}(A(T)))\\ &= \mathcal{I}^*(\{\langle\langle x,t\rangle, \inf_{y\in E}\mu_{A(T)}(y,t), \sup_{y\in E}\nu_{A(T)}(y,t)\rangle|\langle x,t\rangle \in E\times T\})\\ &= \{\langle\langle x,t\rangle, \inf_{u\in T}\inf_{y\in E}\mu_{A(T)}(y,u), \sup_{u\in T}\sup_{y\in E}\nu_{A(T)}(y,u)\rangle|\langle x,t\rangle \in E\times T\}\\ &= \mathcal{I}(\{\langle\langle x,t\rangle, \inf_{u\in T}\mu_{A(T)}(x,u), \sup_{u\in T}\nu_{A(T)}(x,u)\rangle|\langle x,t\rangle \in E\times T\})\\ &= \mathcal{I}(\mathcal{I}^*(A(T))).\end{aligned}$$

This completes the proof.

Intuitionistic fuzzy feeble topological structures containing intuitionistic fuzzy temporal operators

In some previous research, the concept of a Feeble MTS (FMTS) is discussed. Here, we will modify it in the case of the TTSs.
Let for $1 \leq s \leq 4, 1 \leq t \leq 3$: $J_1, \dots J_s \in \{C1, C2, C3, C4, I1, I2, I3, I4\}$, $K_1, \dots, K_t \in \{C1, C2, C3, I1, I2, I3\}$, R, $S_1, \dots, S_s$, $T_1, \dots, T_t, \in \{\subseteq, \supseteq\}$.
The object $\langle \mathcal{P}(X), \mathcal{E}, \zeta, *, \eta \rangle$ is a R-Feeble $(\tau, \eta; K_1T_1, \dots K_tT_t)$-Temporal $(\varphi; J_1S_1, \dots, J_sS_s)$- Topological Structure (with abbreviation: R-F$(\tau, \eta; K_1T_1, \dots K_tT_t)$-T$(\varphi; J_1S_1, \dots, J_sS_s)$-TS) over the set X, where $\mathcal{E} \in \{\mathcal{O}, \mathcal{Q}\}$ is a topological operator from φ-type generated by operation $\zeta \in \{\Delta, \nabla\}$ and $* \in \{\circ, \bullet\}$ is a temporal operator from τ-type related to operation $\eta \in \{\Delta, \nabla\}$, where $\varphi, \tau \in \{cl, in\}$, so that each one of the both operators (the topological and the temporal) must satisfy

the respective C- or the respective I-conditions and in condition J_i the original relation from the definition of the TTSs is replaced by the relation S_i, or this condition is omitted ($1 \leq i \leq s$) and in condition K_j the original relation is replaced by the relation T_j or this condition is omitted ($1 \leq j \leq t$). Similarly, by $R \in \{\subseteq, \supseteq\}$ we denote the fact that in equality (*) the symbol $=$ is replaced by the relation R.
Below, we will discuss all modifications of Theorems 1 and 2 and for each one of these modifications, we will show that it is FTTS in the above sense.
In a previous research of me, the following assertion is formulated and proved as an IFTTS, while in the sense of the above definition for FTTS, it obtains the form:

Theorem

For each universe E and each time-scale T, $\langle \mathcal{P}(E^*(T)), \mathcal{C}, \cup, \mathcal{C}^*, \cap \rangle$ is an IFF$(cl, \cap; 1 \subseteq)$-T$(cl, \cup)$-TS.

Proof. The validity of the conditions C1–C4 for the topological operator $\mathcal{C}$, of the conditions C2–C4 for the temporal operator $\mathcal{C}^*$ and of condition (*) are checked in Theorem 1. To check condition C1 for the temporal operator $\mathcal{C}^*$, we will first prove that for each $x \in E$:

$$\sup_{u \in T} \min(\mu_{A(T)}(x,u), \mu_{B(T)}(x,u)) \leq \min(\sup_{u \in T} \mu_{A(T)}(x,u), \sup_{u \in T} \mu_{B(T)}(x,u)), \tag{1}$$

$$\inf_{u \in T} \max(\nu_{A(T)}(x,u), \nu_{B(T)}(x,u)) \geq \max(\inf_{t \in T} \nu_{A(T)}(x,t), \inf_{t \in T} \nu_{B(T)}(x,t)). \tag{2}$$

Let for a fixed $x \in E$:

$$\min(\mu_{A(T)}(x,u), \mu_{B(T)}(x,u)) = g(u).$$

Therefore, for each $x \in E$:

$$g(u) \leq \mu_{A(T)}(x,u),$$
$$g(u) \leq \mu_{B(T)}(x,u).$$

Therefore,

$$\sup_{u \in T} g(u) \leq \sup_{u \in T} \mu_{A(T)}(x,u),$$
$$\sup_{u \in T} g(u) \leq \sup_{u \in T} \mu_{B(T)}(x,u)$$

and hence

$$\sup_{u \in T} g(u) \leq \min(\sup_{u \in T} \mu_{A(T)}(x,u), \sup_{u \in T} \mu_{B(T)}(x,u)),$$

Back to the proof of C1, using (1) and (2), we see that:

$$
\begin{aligned}
&\mathcal{C}^*(A(T) \cap B(T)) \\
&= \mathcal{C}^*(\{\langle x, \min(\mu_{A(T)}(x,t), \mu_{B(T)}(x,t)), \\
&\qquad \max(\nu_{A(T)}(x,t), \nu_{B(T)}(x,t))|\rangle \langle x,t\rangle \in E \times T\}) \\
&= \{\langle x, \sup_{u \in T} \min(\mu_{A(T)}(x,u), \mu_{B(T)}(x,u))), \\
&\qquad \inf_{u \in T} \max(\nu_{A(T)}(x,u), \nu_{B(T)}(x,u))\rangle | \langle x,t\rangle \in E \times T\} \\
&\subseteq \{\langle x, \min(\sup_{t \in T} \mu_{A(T)}(x,t), \sup_{t \in T} \mu_{B(T)}(x,t)), \\
&\qquad \max(\inf_{t \in T} \nu_{A(T)}(x,t), \inf_{t \in T} \nu_{B(T)}(x,t)\rangle | \langle x,t\rangle \in E \times T\} \\
&= \{\langle x, \sup_{t \in T} \mu_{A(T)}(x,t), \inf_{t \in T} \nu_{A(T)}(x,t)\rangle | \langle x,t\rangle \in E \times T\} \\
&\qquad \cap \{\langle x, \sup_{t \in T} \mu_{B(T)}(x,t), \inf_{t \in T} \nu_{B(T)}(x,t)\rangle | \langle x,t\rangle \in E \times T\} \\
&= \mathcal{C}^*(A(T)) \cap \mathcal{C}^*(B(T)).
\end{aligned}
$$

Theorem

For each universe E and each time-scale T, $\langle \mathcal{P}(E^*(T)), \mathcal{C}, \cup, \mathcal{I}^*, \cup \rangle$ is an IF$\supseteq$-F$(in, \cup; 1 \supseteq)$-T$(cl, \cup)$-TS.

Proof. The validity of the conditions C1–C4 for the topological operator $\mathcal{C}$ and of the conditions I2–I4 for the temporal operator $\mathcal{I}^*$ are checked in Theorems 1 and 2. Now, we must check condition C1 for the temporal operator $\mathcal{I}^*$ and condition $(*)$. The check of the condition C1 uses the inequalities (1) and (2):

$$
\begin{aligned}
&\mathcal{I}^*(A(T) \cup B(T)) \\
&= \mathcal{I}^*(\{\langle\langle x,t\rangle, \max(\mu_{A(T)}(x,t), \mu_{B(T)}(x,t)), \\
&\qquad \min(\nu_{A(T)}(x,t), \nu_{B(T)}(x,t))|\rangle \langle x,t\rangle \in E \times T\}) \\
&= \{\langle\langle x,t\rangle, \inf_{u\in T} \max(\mu_{A(T)}(x,u), \mu_{B(T)}(x,u))), \\
&\qquad \sup_{u\in T} \min(\nu_{A(T)}(x,u), \nu_{B(T)}(x,u))\rangle | \langle x,t\rangle \in E \times T\} \\
&\supseteq \{\langle\langle x,t\rangle, \max(\inf_{u\in T} \mu_{A(T)}(x,u), \inf_{u\in T} \mu_{B(T)}(x,u)), \\
&\qquad \min(\sup_{u\in T} \nu_{A(T)}(x,u), \sup_{u\in T} \nu_{B(T)}(x,u)\rangle | \langle x,t\rangle \in E \times T\} \\
&= \{\langle\langle x,t\rangle, \inf_{u\in T} \mu_{A(T)}(x,u), \sup_{u\in T} \nu_{A(T)}(x,u)\rangle | \langle x,t\rangle \in E \times T\} \\
&\qquad \cup \{\langle\langle x,t\rangle, \inf_{u\in T} \mu_{B(T)}(x,u), \sup_{u\in T} \nu_{B(T)}(x,u)\rangle | \langle x,t\rangle \in E \times T\} \\
&= \mathcal{I}^*(A(T)) \cup \mathcal{I}^*(B(T)).
\end{aligned}
$$

For the check of the condition (*), first, we will prove that:

$$\inf_{u\in T}\sup_{y\in E}(\mu_{A(T)}(y,u),\mu_{B(T)}(y,u)) \geq \sup_{y\in E}(\inf_{u\in T}\mu_{A(T)}(y,u),\inf_{u\in T}\mu_{B(T)}(y,u)), \tag{3}$$

$$\sup_{u\in T}\inf_{y\in E}(\nu_{A(T)}(y,u),\nu_{B(T)}(y,u)) \leq \inf_{y\in E}(\sup_{t\in T}\nu_{A(T)}(y,t),\sup_{t\in T}\nu_{B(T)}(y,t)). \tag{4}$$

Let for a fixed $u \in T$:

$$\sup_{y\in E}(\mu_{A(T)}(y,u),\mu_{B(T)}(y,u)) = h(u).$$

Therefore, for each $y \in E$:

$$h(u) \geq \mu_{A(T)}(y,u),$$

$$h(u) \geq \mu_{B(T)}(y,u).$$

Therefore,

$$\inf_{u \in T} h(u) \geq \inf_{u \in T} \mu_{A(T)}(y,u),$$
$$\inf_{u \in T} h(u) \geq \inf_{u \in T} \mu_{B(T)}(y,u)$$

and hence

$$\inf_{u \in T} h(u) \geq \sup_{y \in E}(\inf_{u \in T} \mu_{A(T)}(y,u), \inf_{u \in T} \mu_{B(T)}(y,u)),$$

i.e., (1) is valid. Proving (2) is done in a similar manner.

Back to the proof of (*), using (3) and (4), we see that:

$$\mathcal{I}^*(\mathcal{C}(A(T)))$$
$$= \mathcal{I}^*(\{\langle\langle x, t\rangle, \sup_{y\in E} \mu_{A(T)}(y,t), \inf_{y\in E} \nu_{A(T)}(y,t)\rangle | \langle x, t\rangle \in E \times T\})$$
$$= \{\langle\langle x, t\rangle, \inf_{u\in T} \sup_{y\in E} \mu_{A(T)}(y,u), \sup_{u\in T} \inf_{y\in E} \nu_{A(T)}(y,u)\rangle | \langle x, t\rangle \in E \times T\})$$
$$\supseteq \{\langle x, \sup_{y\in E} \inf_{u\in T} \mu_{A(T)}(y,u), \inf_{y\in E} \sup_{u\in T} \nu_{A(T)}(y,u)\rangle | \langle x, t\rangle \in E \times T\}$$
$$= \mathcal{C}(\{\langle x, \inf_{t\in T} \mu_{A(T)}(x,t), \sup_{t\in T} \nu_{A(T)}(x,t)\rangle | \langle x, t\rangle \in E \times T\})$$
$$= \mathcal{C}(\mathcal{I}^*(A(T))).$$

This completes the proof.

In the same manner we can prove the following assertions, too.

Theorem

For each universe E and each time-scale T, $\langle \mathcal{P}(E^*(T)), \mathcal{C}, \cup, \mathcal{I}^*, \cap \rangle$ is an IF$\supseteq$-F$(in, \cap)$-T$(cl, \cup)$-TS.

Theorem

For each universe E and each time-scale T, $\langle \mathcal{P}(E^*(T)), \mathcal{C}, \cap, \mathcal{C}^*, \cup \rangle$ is an IFFcl-T$(cl, \cap; 1 \subseteq)$-TS.

Theorem

For each universe E and each time-scale T, $\langle \mathcal{P}(E^*(T)), \mathcal{C}, \cap, \mathcal{C}^*, \cap \rangle$ is an IFF$(cl, \cap; 1 \subseteq)$-T$(cl, \cap; 1 \subseteq)$-TS.

Theorem

For each universe E and each time-scale T, $\langle \mathcal{P}(E^*(T)), \mathcal{C}, \cap, \mathcal{I}^*, \cup \rangle$ is an IF$\subseteq$-F$(in, \cup; 1 \supseteq)$-T$(cl, \cap; 1 \subseteq)$-TS.

Theorem

For each universe E and each time-scale T, $\langle \mathcal{P}(E^*(T)), \mathcal{C}, \cap, \mathcal{I}^*, \cap \rangle$ is an IF$\subseteq$-F$(in, \cup)$-T$(cl, \cap; 1 \subseteq)$-TS.

Theorem

For each universe E and each time-scale T, $\langle \mathcal{P}(E^*(T)), \mathcal{I}, \cup, \mathcal{C}^*, \cup \rangle$ is an IF$\supseteq$-F$(cl, \cup)$-T$(in, \cup; \supseteq)$-TS.

Theorem

For each universe E and each time-scale T, $\langle \mathcal{P}(E^*(T)), \mathcal{I}, \cup, \mathcal{C}^*, \cap \rangle$ is an IF$\supseteq$-F$(cl, \cap; 1 \subseteq)$-T$(in, \cup; \supseteq)$-TS.

Theorem

For each universe E and each time-scale T, $\langle \mathcal{P}(E^*(T)), \mathcal{I}, \cup, \mathcal{I}^*, \cup \rangle$ is an IFF$(in, \cup; 1 \supseteq)$-T$(in, \cup; \supseteq)$-TS.

Theorem

For each universe E and each time-scale T, $\langle \mathcal{P}(E^*(T)), \mathcal{I}, \cup, \mathcal{I}^*, \cap \rangle$ is an IFF$(in, \cap)$-T$(in, \cup; \supseteq)$-TS.

Theorem

For each universe E *and each time-scale* T, $\langle \mathcal{P}(E^*(T)), \mathcal{I}, \cap, \mathcal{C}^*, \cup \rangle$ *is an IF*$\supseteq$*-F*$(cl, \cup)$*-T*$(in, \cap)$*-TS.*

Theorem

For each universe E *and each time-scale* T, $\langle \mathcal{P}(E^*(T)), \mathcal{I}, \cap, \mathcal{C}^*, \cap \rangle$ *is an IF*$\supseteq$*-F*$(cl, \cap)$*-T*$(in, \cap)$*-TS.*

Theorem

For each universe E *and each time-scale* T, $\langle \mathcal{P}(E^*(T)), \mathcal{I}, \cap, \mathcal{I}^*, \cup \rangle$ *is an IFF*$(in, \cup; 1 \supseteq)$*-T*$(in, \cap)$*-TS.*

Conclusion

In the paper we described all TTSs generated by the IFTOs $\mathcal{C}^*$ and $\mathcal{I}^*$. In a next research, we will discuss other structures in which the IFTOs will play the role of the topological operators and these structures will contain modal operators.

Thank you for attention!

The present research has been supported by the Bulgarian National Science Fund under Grant Ref. No. KP-06-N72/8 from 14.12.2023.