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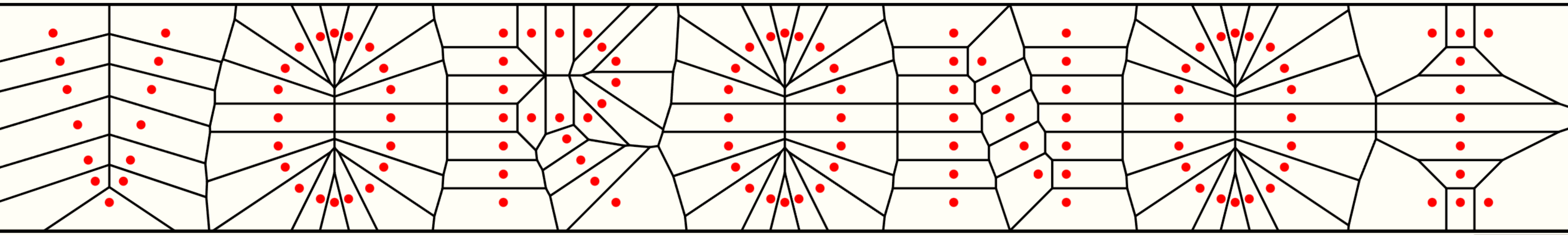

Diagram Partitioning of the Intuitionistic Fuzzy Interpretational Triangle

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Remarks on Intuitionistic Fuzziness

Let a universe set E be fixed. An IFS $A \subseteq E$ is called intuitionistic fuzzy set if

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in E\},$$

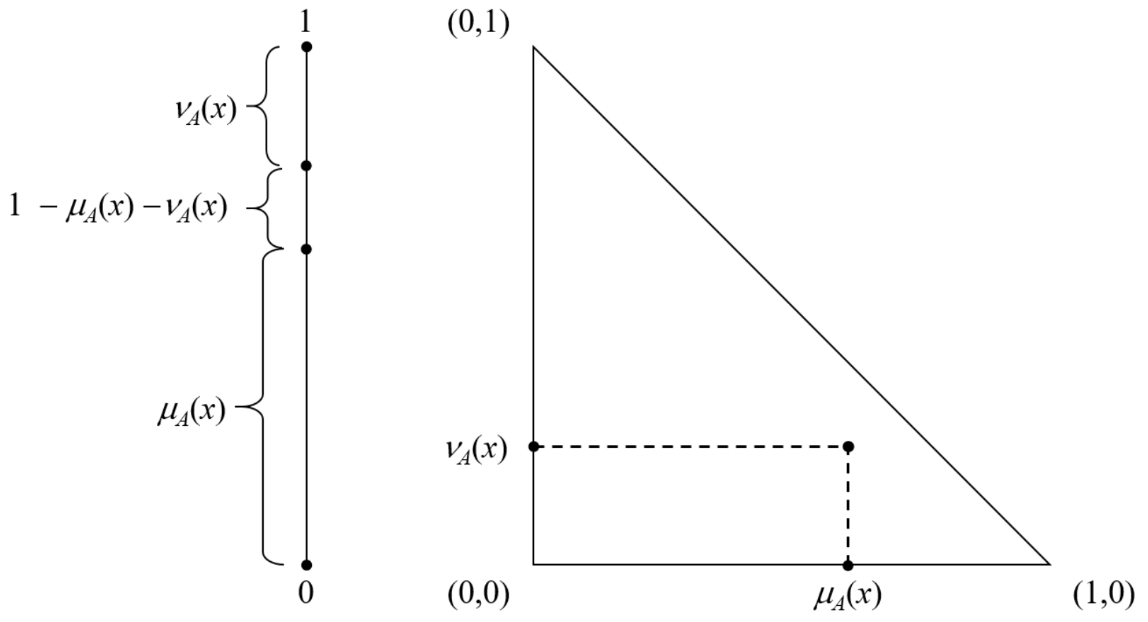
where the functions $\mu_A : E \rightarrow [0, 1]$ and $\nu_A : E \rightarrow [0, 1]$ define the degree of membership and the degree of non-membership of the element $x \in E$ to A, respectively, so that $0 \leq \mu_A(x) + \nu_A(x) \leq 1$.

In case of $\mu_A(x) + \nu_A(x) < 1$, this is the case when a third complementary degree occurs, $\pi A(x) = 1 - \mu_A(x) - \nu_A(x)$, and this is the degree of uncertainty regarding the element x's belonging to the set A.

When the degree $\pi_A(x) = 0$, we obtain the boundary case coinciding with the ordinary Zadeh's fuzzy sets.

Remarks on Intuitionistic Fuzziness

IFSs have numerous different geometrical interpretations, but the two standard ones are shown in Figure 1, where the linear one to the left is in the tradition of visualizing fuzzy sets, and the planar one to the right is the so called intuitionistic fuzzy interpretational triangle (IFIT). It is considered the most representative for the concept, and one that has no analogue in the ordinary fuzzy sets.



Remarks on Intuitionistic Fuzziness

It is well-known that the interval [0, 1] plays the role of a universal representative of any interval from the real line. This is the reason that Lotfi Zadeh used it to define the set of membership values for the elements of an arbitrary fuzzy set.

Here, for the first time we will discuss the capacity of the IFIT to represent any set of points in the finite plane. For this aim, we will formulate and prove the following assertion.

Theorem 1. *Let us have a finite or infinite set of points in a finite plane. There is a bijective function that transforms these points into points in the IFIT.*

Remarks on Intuitionistic Fuzziness

Theorem 1. *Let us have a finite or infinite set of points in a finite plane. There is a bijective function that transforms these points into points in the IFIT.*

Proof. Let a finite plane be given and let it contain either a finite or an infinite set of points. As the plane is finite by definition, therefore, in the orthogonal coordinate system introduced in it, there can be constructed four lines with equations

$$l_1 : x = a,\ l_2 : x = b,\ l_3 : y = c,\ l_4 : y = d,$$

where $a < b$ and $c < d$, and the coordinates of each one of these points, e.g., $x = (x_1, x_2)$ satisfy the inequalities $a \leq x_1 \leq b$, $c \leq x_2 \leq d$.

Now, we define the bijective function

$$G(x_1, x_2) = \left(\frac{x_1 - a}{b - a}, \frac{x_2 - c}{d - c} \right).$$

Remarks on Intuitionistic Fuzziness

For it, we see immediately that $G(a, c) = (0, 0)$, denoted by $\boldsymbol{U}$ (for "Uncertainty"), $G(a, d) = (0, 1)$, denoted by $\boldsymbol{F}$ (for "Falsity"), $G(b, c) = (1, 0)$, denoted by $\boldsymbol{T}$ (for "Truth"), and $G(b, d) = (1, 1)$, and any arbitrary point $x = (x_1, x_2)$ will lie in the unit square with coordinates $(0, 0)$, $(1, 0)$, $(1, 1)$, and $(0, 1)$.

Now, using the formula

$$g(x,y) = \begin{cases} (0,0), & \text{if } x = y = 0, \\ \left(\dfrac{x^2}{x+y}, \dfrac{xy}{x+y}\right), & \text{if } x, y \in [0,1] \text{ and } x \geq y \text{ and } x + y > 0 \\ \left(\dfrac{xy}{x+y}, \dfrac{y^2}{x+y}\right), & \text{if } x, y \in [0,1] \text{ and } x \leq y \text{ and } x + y > 0 \end{cases}$$

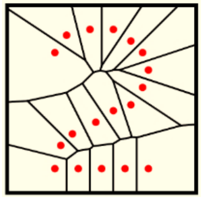
we transform bijectively the unit square together with its adjacent points to the intuitionistic fuzzy interpretational triangle, which proves the Theorem.

Remarks on Intuitionistic Fuzziness

Corollary 1. *Each planar graph (and in the particular case, lattice), lying in a finite plane can be transformed bijectively to the intuitionistic fuzzy interpretational triangle.*

Corollary 2. *Each L-fuzzy sets, where L is a lattice in a finite plane, can be represented as an IFS.*

The formulated and proved Theorem 1 is necessary for the modification of the Voronoi method which will be proposed now, as inspired by intuitionistic fuzziness.



Modification of the Voronoi method inspired by IFs

For our newly proposed method we draw inspiration from the intuitionistic fuzziness, but our approach significantly differs from the earlier attempts for constructing IF Voronoi diagrams, as described in the works of Todorova *et al.* [5, 6].

Let us have a universe E, which for our purpose—and in line with our above formulated Theorem—is ***not the infinite plane*** but limited to the intuitionistic fuzzy interpretational triangle (IFIT), and let us have an IFS X over it.

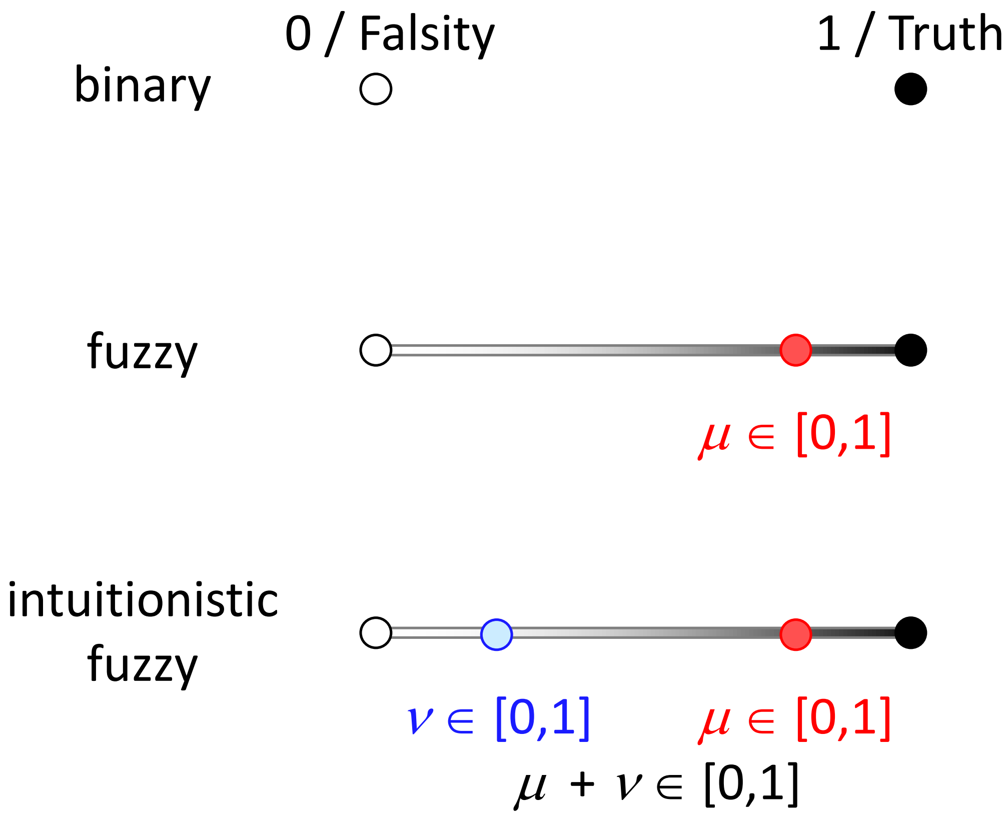
Let the IFIT, including its sides and vertices, contain the points that correspond to the elements of X with their coordinates being the values of the membership and non-membership functions with which the elements belong and not belong to the set.

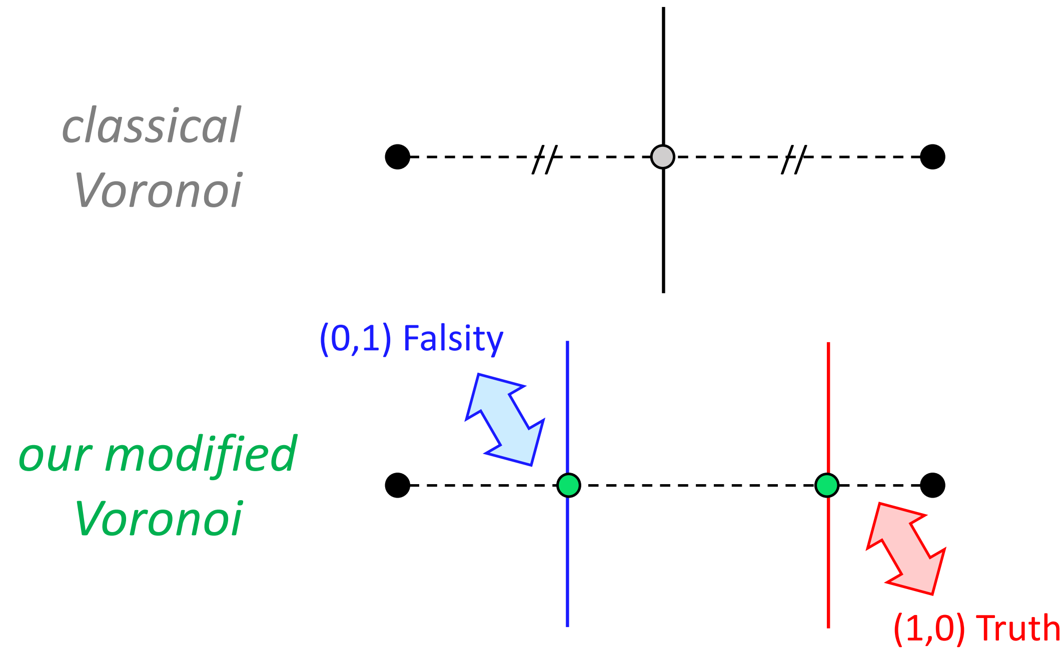
[5] Todorova, L., Antonov, A., & Hadjitodorov, S. (2004). Intuitionistic fuzzy Voronoi diagrams – Definition and properties. *Notes on Intuitionistic Fuzzy Sets*, 10(4), 56–60.

[6] Todorova, L., & Antonov, A. (2005). Intuitionistic fuzzy Voronoi diagrams – Definition and properties. Part 2. *Notes on Intuitionistic Fuzzy Sets*, 11(3), 88–90.

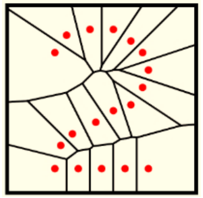
Modification of the Voronoi method inspired by IFs

The inspiration for our approach is deeply rooted in the concept of IFSs and their visual representation.





The segment between two seed points is divided by ***two*** *perpendiculars, whose precise locations depend on both the seed points' locations and the IFIT's vertices.*



Modification of the Voronoi method inspired by IFs

In the classical Voronoi construction, the tessellation depends **only on the seed points**.

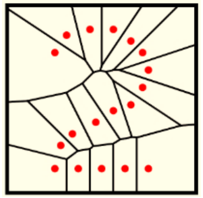
The **surrounding space has no influence**.

In our modification of the Voronoi construction:

1. Space will be limited to the IFIT,
2. Tesselation will be influenced not only by the seed points themselves, but also by the semantic structure of the IFIT and

   – explicitly – by the concepts of truth and falsity.

   – implicitly – by the concept of complementary uncertainty.



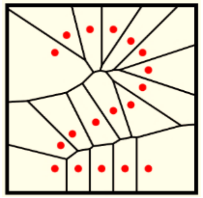
Modification of the Voronoi method inspired by IFs

Each seed point is first characterized by its relationship to two reference points – namely, the IF constants “Falsity” (0,1) and “Truth” (1,0). These vertices:

- are **not additional** seed points,
- **nor simply boundary** vertices of the space,
- serve as **geometric reference points** that orient the whole construction.

Although the “Uncertainty” vertex ***U*** (0,0) does not participate directly in the pairwise assignment of the seed points in the same manner as the “Truth” and “Falsity” vertices, its semantic role is nevertheless reflected in the construction:

- complementary to ***F*** and ***T*** that guide the construction of the IF Voronoi partition,
- represented implicitly by the hesitation regions that emerge naturally between neighbouring IF Voronoi cells in our modified method.



Modification of the Voronoi method inspired by IFs

ALGORITHM

Input: Seed points $P_1, P_2, \ldots, P_n$.

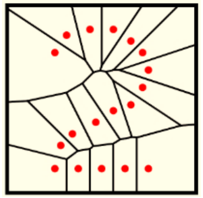
Output: Intuitionistic fuzzy (IF) Voronoi regions $R_1, R_2, \ldots, R_n$.

1. Compute the Euclidean distances $d_F(P_i)$ and $d_T(P_i)$ for all seed points P_i.

2. For each unordered pair (P_i, P_j), repeat:
   - 2.1 If $d_F(P_i) \leq d_F(P_j)$, associate P_i with F and P_j with T; otherwise, associate P_i with T and P_j with F. (Tie-breaking rule: F).
   - 2.2 Construct the points Q_{ij} and Q_{ji} on the segment P_iP_j using the Angle Bisector Theorem with respect to their associated reference vertices.
   - 2.3 Construct the pair of IF perpendiculars passing through Q_{ij} and Q_{ji}, respectively, and perpendicular to the segment P_iP_j.

3. For each seed point P_i, repeat:
   - 3.1 Select, for each neighbouring seed point P_j, the IF perpendicular adjacent to P_i.
   - 3.2 Intersect the corresponding half-planes.
   - 3.3 Obtain the IF Voronoi region R_i.

4. Return the set of IF Voronoi regions.



Modification of the Voronoi method inspired by IFs

Remarks on the Algorithm:

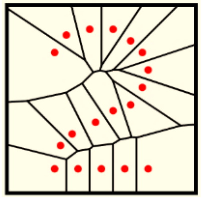
- The algorithm alternates between **local** and **global** computations:
  - Step 1: local to each seed point (compute its distances).
  - Step 2: local to each pair of seed points (construct Q_{ij}, Q_{ji}, and the IF perpendiculars).
  - Step 3: global to each seed point (assemble its region from all pairwise constructions).

  Nice computational hierarchy:

$$\text{point} \rightarrow \text{pair} \rightarrow \text{region}$$

- Computational complexity:

$$O(n) + O(n^2) + O(n^2) = O(n^2)$$



Modification of the Voronoi method inspired by IFs

Remarks on the software implementation:

- HTML5
- CSS3
- Vanilla JavaScript (ES6)
- SVG graphics
- Client-side web application
- No external libraries or frameworks

Development approach:

- Iterative human–AI collaborative design / Research-driven vibe coding
- ChatGPT 5.5 Pro (OpenAI)
- More than 100 iterative human–AI design cycles

Intuitionistic Fuzzy Voronoi Toolkit: **Demo**

Test it yourself!

Desktop and mobile

.html, 27 Kb

https://bit.ly/if-voronoi

Discussion

The presented method bears some similarity with existing attempts of fuzzifying the Voronoi tessellation found in the current literature, e.g. [8–11].

There is however a notable difference.

In our method of computing the locations of pairs of perpendicular Voronoi edges, the resultant edges are not equidistant of the seed points and of the standard bisector, as in the classical Voronoi method and in the existing fuzzified and approximate versions of the Voronoi diagram.

[8] Arotaritei, D.: A method to construct approximate fuzzy Voronoi diagram for fuzzy numbers of dimension two. *International Journal of Computers Communications & Control* **9**(4), 389–396 (2014)

[9] Dai, M., Dong, F., Hirota, K.: Fuzzy three-dimensional Voronoi diagram and its application to geographical data analysis. *Journal of Advanced Computational Intelligence and Intelligent Informatics* **16**(2), 191–198 (2012)

[10] Arotaritei, D., Ionescu, F.: Fuzzy Voronoi diagram for disjoint fuzzy numbers of dimension two. *Journal of Intelligent & Fuzzy Systems* **26**, 1253–1262 (2014)

[11] Kiseleva, E., Prytomanova, O., Hart, L., Blyuss, O.: Application of the theory of optimal set partitioning for constructing fuzzy Voronoi diagrams. In: Zgurovsky, M., Pankratova, N. (eds.) *System Analysis & Intelligent Computing*, SAIC 2020, Studies in Computational Intelligence, vol. 1022, pp. 287–313. Springer (2022)

Discussion

The presented modified Voronoi method in the IFIT can be applied to a situation when the IFIT needs to be tessellated, so that regions of proximity are formed near specific points or more specifically to the vertices that represent the complete Truth and Falsity in the intuitionistic fuzzy scenario.

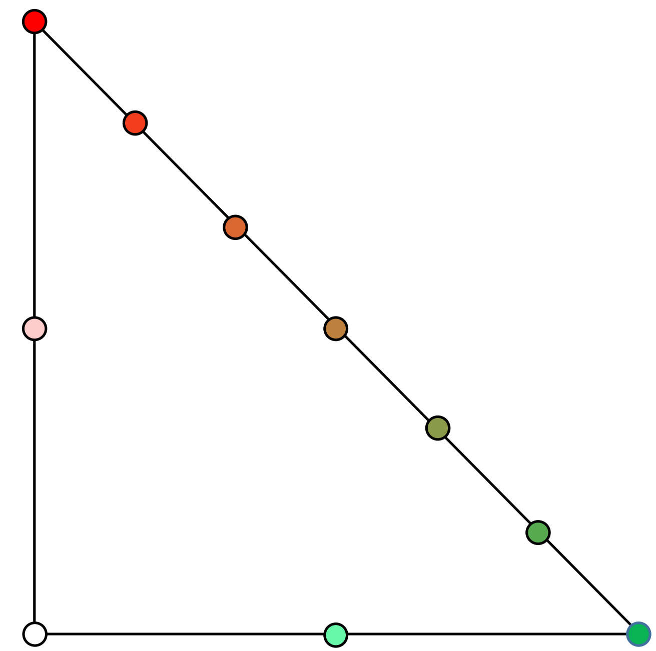
For instance, for a large set of IFS elements, the modified Voronoi method a smaller number of more representative points (e.g. centers of clusters, etc.), and thus the produced tessellation, specifically in the case when well expressed “belts” are formed in between the regions, can be of support for the particular decision making problems on both the level of individual points and the so formed regions, depending on their proximity to these IF constants.

Discussion

In this sense, we can consider a further generalization of our modification of the Voronoi tessellation algorithm, which uses other user defined IF values (points within the IFIT) instead of the IFIT vertices ***F*** and ***T***.

For example, among the reasonable alternatives, we may consider

- vertex ***U***,
- the midpoints of the IFIT sides,
- or a set of equidistant points plotted along the IFIT's hypotenuse which represent a scale of gradually varying degrees of membership and non-membership with zero uncertainty.



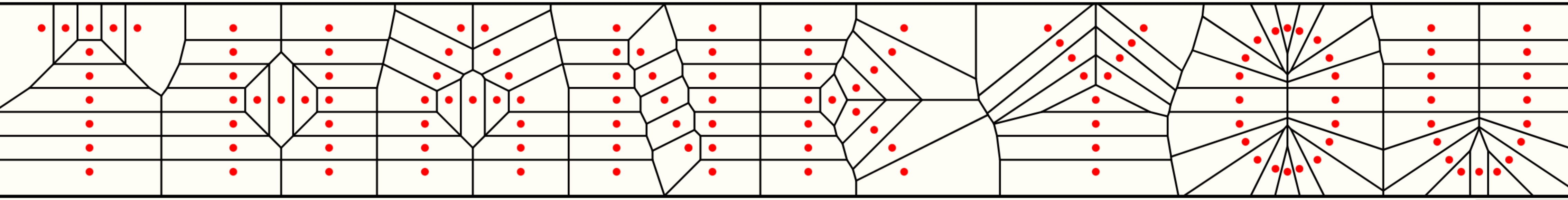
Discussion

Regarding the application of the method, we envisage that it could be well applicable to the intuitionistic fuzzy sets-based decision support method of InterCriteria Analysis (ICA), as well as to other decision making procedures where intuitionistic fuzziness is involved and interpretation of the IF values as points within the IFIT can be an integral part of the analysis.

Particularly in the context of ICA, the proposed method suggests a new geometrically motivated approach to determining the threshold values, against which the computed InterCriteria pairs' levels are to be compared in order to estimate if the criteria in the pair exhibit:

- a level of positive consonance (indicative of presence of correlation between two criteria),
- a level of negative consonance (indicative of the lack of correlation),
- or a level of dissonance (no correlation between the criteria).

Sincerely



for your attention!

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