

INTUITIONISTIC FUZZY SETS

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A definition of the concept 'intuitionistic fuzzy set' (IFS) is given, the latter being a generalization of the concept 'fuzzy set' and an example is described. Various properties are proved, which are connected to the operations and relations over sets, and with modal and topological operators, defined over the set of IFS's.

Keywords: Intuitionistic fuzzy set, Fuzzy Set, Modal operator (necessity, possibility), Topological operator (interior, closure).

The notions of intuitionistic fuzzy set (IFS) and intuitionistic L-fuzzy sets (ILFS) were introduced in [1, 2] and [3], respectively, as a generalization of the notion of fuzzy set (FS).

Let a set E be fixed. An ILFS A^* in E is an object having the form

$$A^* = \{(x, \mu_A(x), \nu_A(x)) \mid x \in E\},$$

where the functions $\mu_A: E \rightarrow L$ and $\nu_A: E \rightarrow L$ define the degree of membership and the degree of nonmembership of the element $x \in E$ to $A \subseteq E$ (for simplicity we shall write A instead of A^*), respectively, the functions μ_A and ν_A should satisfy the condition:

$$(\forall x \in E)(\mu_A(x) \leq N(\nu_A(x))),$$

where $N: L \rightarrow L$ is an involutive order reversing operation in the lattice $(L, \leq)$. When $L = [0, 1]$, the object A is an IFS and the following condition holds:

$$(\forall x \in E)(0 \leq \mu_A(x) + \nu_A(x) \leq 1).$$

Obviously every FS has the form $\{(x, \mu_A(x), 1 - \mu_A(x)) \mid x \in E\}$.

Let us denote everywhere for simplicity by $\{(x, \mu_A(x)) \mid x \in E\}$ the set $\{(x, \mu_A(x), 1 - \mu_A(x)) \mid x \in E\}$. The definition makes clear that for the so constructed new type of FS the logical law of the excluded middle is not valid, similarly to the case in intuitionistic mathematics. Herefrom emerges the name of that set. Note that in the case when L is a lattice it is possible to introduce as example about a set for which $\mu_A(x), N(\mu_A(x)) < \sup L$, where $\sup L$ is a maximal element of L , but as in that case $\nu_A(x) = N(\mu_A(x))$, the FS $\{(x, \mu_A(x)) \mid x \in E\}$ cannot be equivalent to the IFS $\{(x, \mu_A(x), \nu_A(x)) \mid x \in E\}$.

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Intuitionistic Fuzzy Sets in Retrospective and Perspective: Forty Years Since Atanassov's Landmark Paper

Before describing the properties of IFS's we shall give an example of an IFS which is not an FS. The example is written after an idea of the author in [4].

Let A, B, C and D be four convex, closed, connected and compact sets in the Euclidean plane, as shown in Figure 1 ($A \cap B = A \cap C = B \cap C = B \cap D = C \cap D = \emptyset$). Let in that plane Cartesian coordinates Ox_1, Ox_2 be given and let the sets P, Q, R, S, T, U and V be respectively their orthogonal projections over the axis Ox_1 . By the real number $l(y, X)$ we denote the length (as regards the length unit introduced for the axis Ox_1) of the segment in the set X lying on a line perpendicular to the axis Ox_1 , passing through point y from Ox_1 . Let $E = A \cup B \cup C \cup D$ be the universe for our further considerations, and let the sets F and G satisfy the following conditions:

- $A \subseteq F \subseteq A \cup C \cup D$;
- $B \subseteq G \subseteq B \cup C \cup D$;
- $F \cap G = \emptyset$;
- $F \cup G \subseteq E$.

From the last two conditions it follows that the set F is strictly included in the supplement of the set G to E , and from the first two - that the trivial cases $F = A$ and $G = B$ are excluded. The four conditions are independent.

Let us have the possibility to observe only the projections of the points from E over the axis Ox_1 , and for every $x \in E$ we know only (observing its projection $y \in P \cup Q \cup R \cup S \cup T \cup U$) the value of $l(y, X)$, where X is some of the sets A, B, C and D .

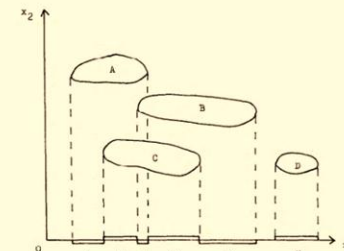


Fig. 1.

Our aim will be to show the form of the functions of the elements of E towards the set F and G in the plane, as shown in Figure

$$\mu_F(x) = \begin{cases} 1 & \text{if } x \in A \\ \frac{l(y, A)}{l(y, A) + l(y, C)} & \text{if } x \in C \\ \frac{l(y, A)}{l(y, A) + l(y, B) + l(y, C)} & \text{if } x \in B \\ 0 & \text{if } x \in D \end{cases}$$

Then

$$\mu_F(x) + \nu_F(x) = \begin{cases} 1 & \text{if } x \in A \\ \frac{l(y, A)}{l(y, A) + l(y, C)} + \frac{l(y, B)}{l(y, A) + l(y, B) + l(y, C)} & \text{if } x \in C \\ \frac{l(y, A)}{l(y, A) + l(y, B) + l(y, C)} + \frac{l(y, B)}{l(y, A) + l(y, B) + l(y, C)} & \text{if } x \in B \\ 1 & \text{if } x \in D \end{cases}$$

i.e. $0 \leq \mu_F(x) + \nu_F(x) \leq 1$.

If $\pi_F(x) = 1 - \mu_F(x) - \nu_F(x)$, then $\pi_F(x)$ is the element $x \in E$ to the set $F \subseteq E$. In the example

$$\pi_F(x) = \begin{cases} 0 & \text{if } x \in A \\ \frac{l(y, C)}{l(y, A) + l(y, C)} & \text{if } x \in C \\ \frac{l(y, C)}{l(y, A) + l(y, B) + l(y, C)} & \text{if } x \in B \\ \frac{l(y, C)}{l(y, B) + l(y, C)} & \text{if } x \in D \\ 1 & \text{if } x \in E \end{cases}$$

Intuitionistic Fuzzy Sets: Historical Context

When and Where

The common misconception that 1986 marks the origin of the notion has been repeatedly reproduced.

However, the evidence of this misconception can be found in the References section of the paper itself.

IFSs were first proposed in a preprint of a conference held in Sofia, Bulgaria on 20–23 June 1983.

Acknowledgements

I would like to express my warm thanks to D. Sassellov, L. Atanassova, G. Gargov and S. Stoeva for valuable discussions on the problems considered here.

References

- [1] K. Atanassov, Intuitionistic fuzzy sets, in: V. Sgurev, Ed., VII ITKR's Session, Sofia, June 1983 (Central Sci. and Techn. Library, Bulg. Academy of Sciences, 1984).
- [2] K. Atanassov and S. Stoeva, Intuitionistic fuzzy sets, in: Polish Symp. on Interval & Fuzzy Mathematics, Poznan (Aug. 1983) 23–26.
- [3] K. Atanassov and S. Stoeva, Intuitionistic L-fuzzy sets, in: R. Trappl, Ed., Cybernetics and Systems Research 2 (Elsevier Sci. Publ., Amsterdam, 1984) 539–540.
- [4] K. Atanassov, Intuitionistic fuzzy relations, in: L. Antonov, Ed., III International School “Automation and Scientific Instrumentation”, Varna (Oct. 1984) 56–57.

Intuitionistic Fuzzy Sets: Historical Context

When and Where

IFSs originated in Bulgaria, which back then was behind the Iron Curtain.

The era was marked with substantial barriers to accessing and publishing in reputable Western scientific journals.

IFSs were first presented at some local research forums in 1983 and 1984 before the publication in *FS&S*.



Intuitionistic Fuzzy Sets: Historical Context

When and Where

Scientific research in this period relied almost entirely on printed publications, long before

- the launch of World Wide Web,
- widespread digitization,
- and accessibility of research literature as we know it today.



Intuitionistic Fuzzy Sets: Historical Context

When and Where

Another important nuance was the author's young age at the time of the concept's creation in 1983. Atanassov, born in 1954, was not yet 30 when he was first inspired by the idea of generalizing fuzzy sets IFSs.

He was not part of any existing "school of thought" in Bulgaria, yet, established one himself.



Intuitionistic Fuzzy Sets: Historical Context

When and Where

Wider recognition of the foundation year 1983 came only after the Scopus-indexed *Int. J. BioAutomation* reprinted in 2016 a set of the earliest preprints on IFs and other Atanassov's research endeavours, including their facsimiles / translations.

The QR code to the right links to the dedicated volume with reprints.



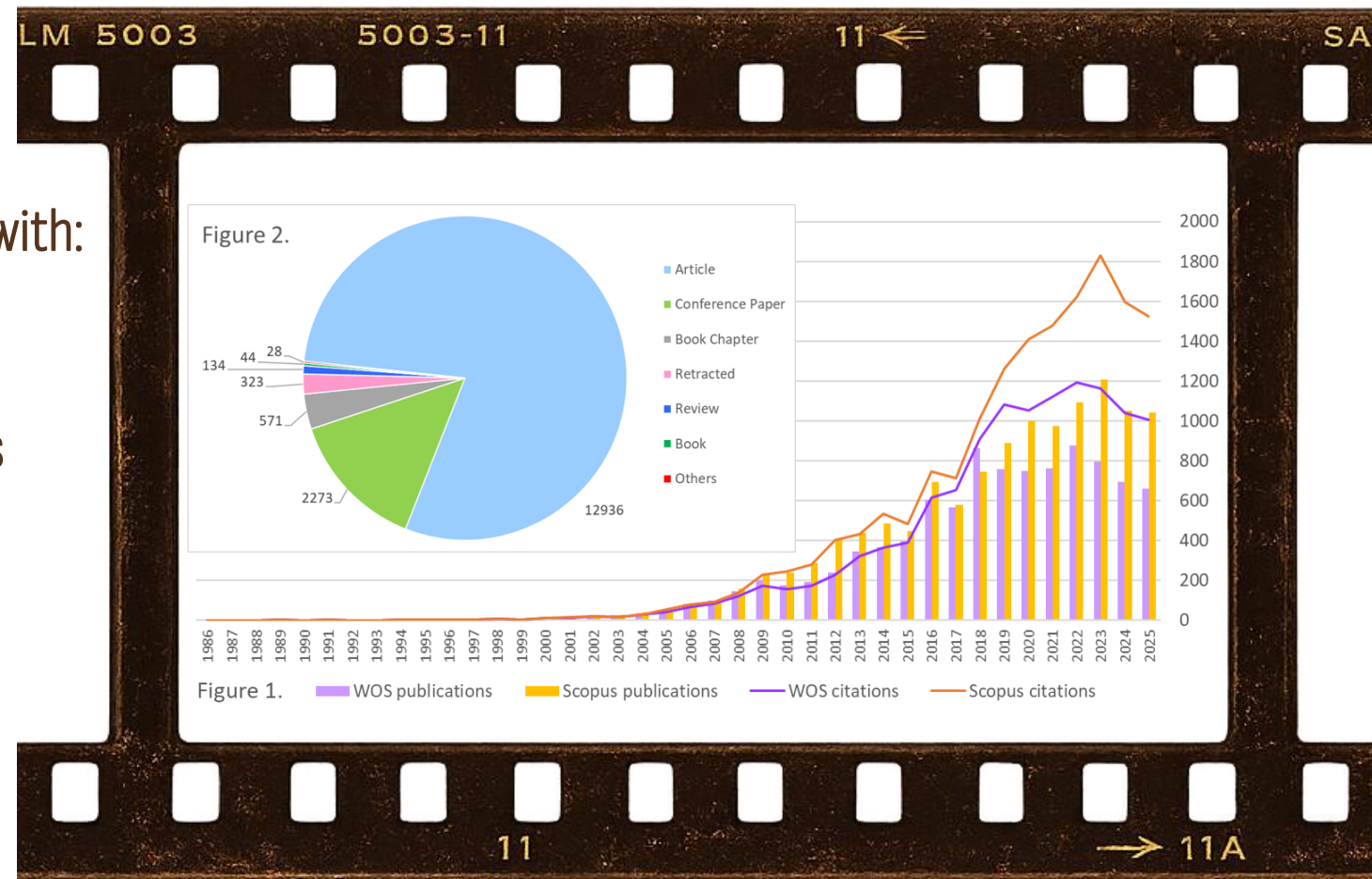
Intuitionistic Fuzzy Sets: 40-Year Legacy in Numbers

Citation Impact

As of 31 Dec 2025, the paper in *FS&S* is the most cited publication on IFSs with:

- > 11 200 citations in WoS Core Collection
- > 13 250 across all WoS databases
- > 16 200 citations in Scopus

As of the start of 21st century, it has become a major catalyst of research in the area of IFSs, and generally fuzzy sets, and their extensions.



Intuitionistic Fuzzy Sets: 40-Years Legacy in Numbers

International Outreach

Early adopters of IFSs:

- 1988 – Romania

Toader Buhaescu (1934 – 2026)



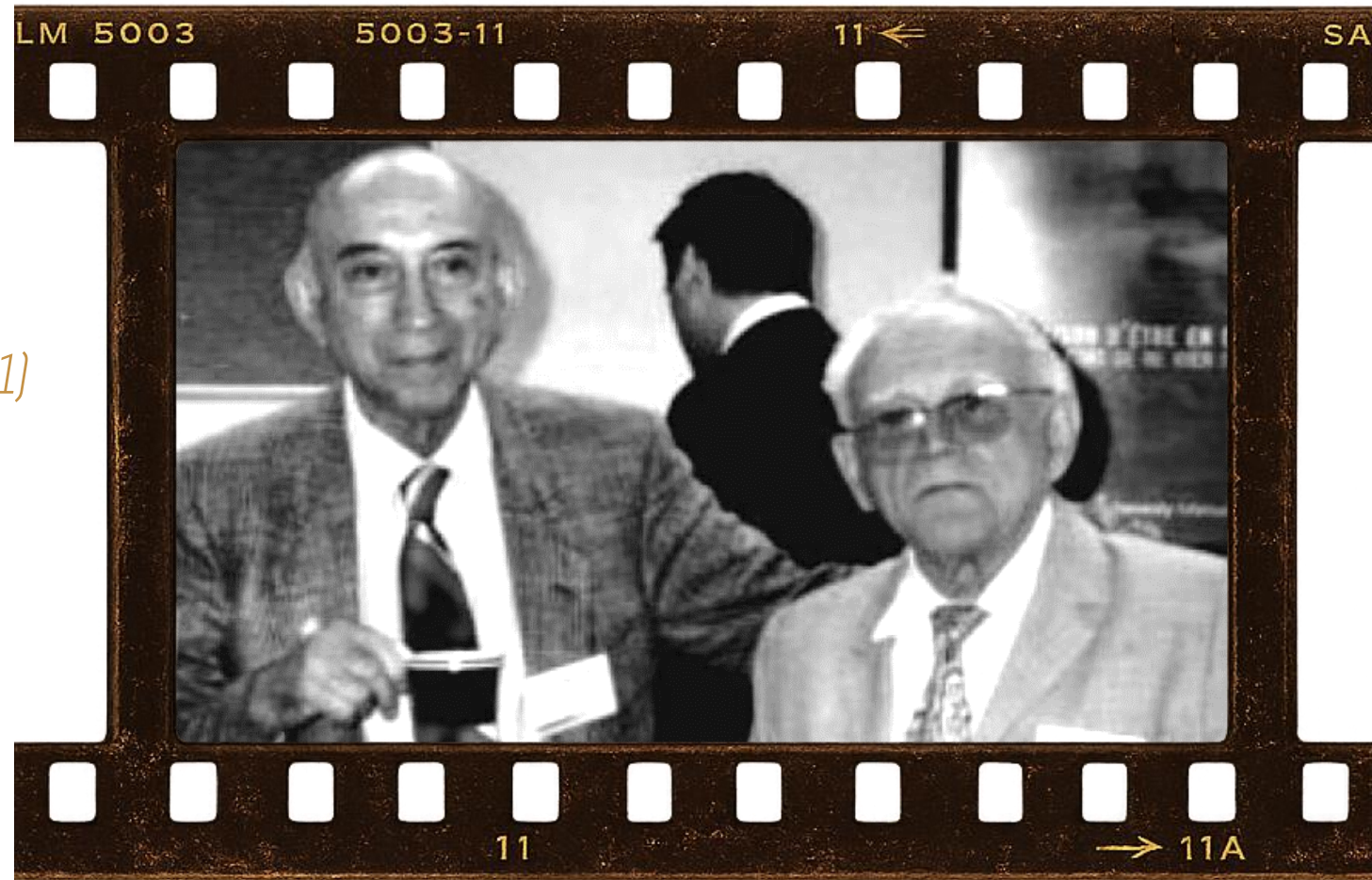
Intuitionistic Fuzzy Sets: 40-Years Legacy in Numbers

International Outreach

Early adopters of IFSs:

- 1988 – Romania
- 1990 – Poland

Tadeusz Gerstenkorn (1927 – 2021)



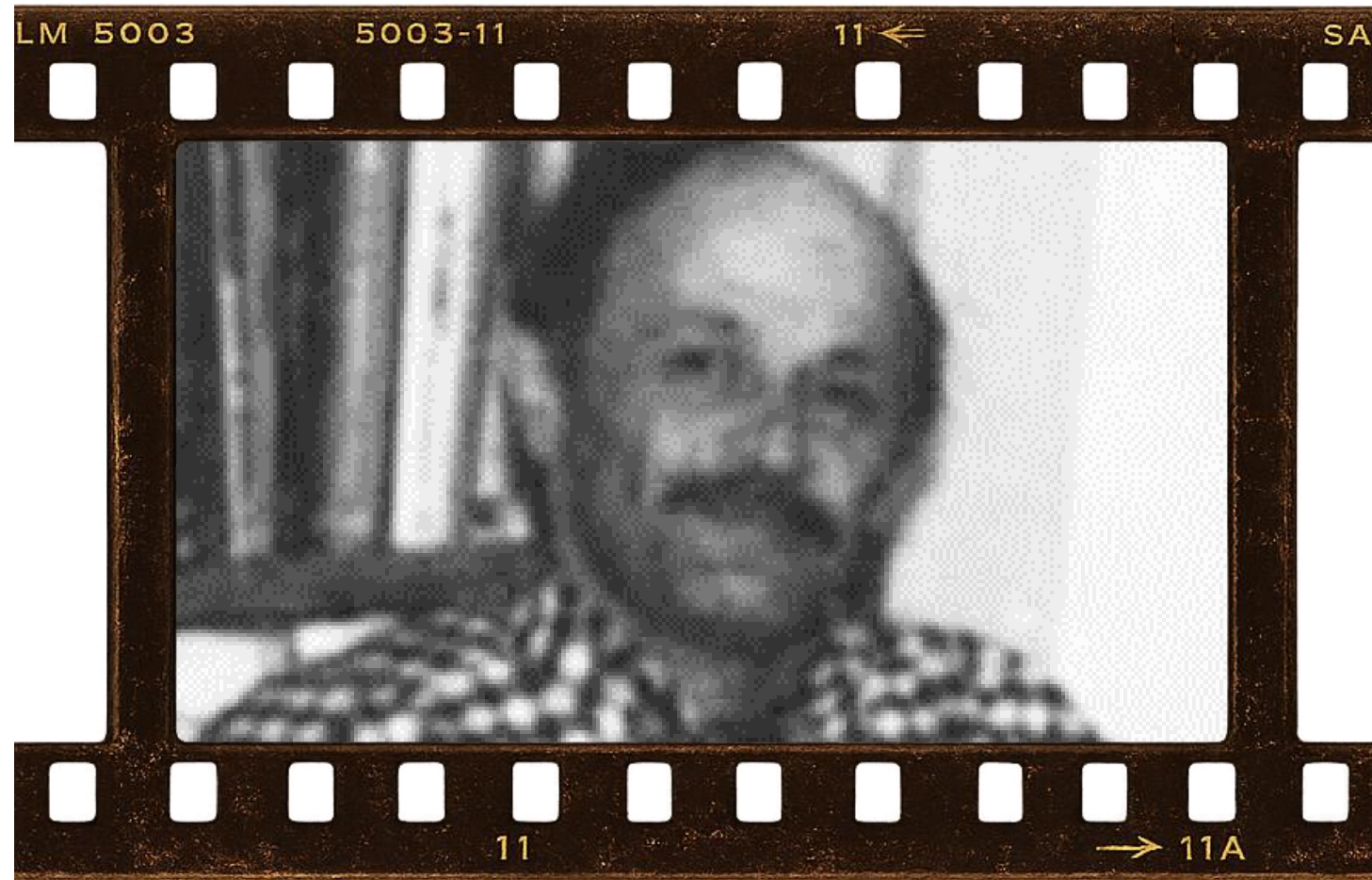
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International Outreach

Early adopters of IFSs:

- 1988 – Romania
- 1990 – Poland
- 1992 – Spain
- 1993 – India and China
- 1995 – Türkiye

Doğan Çoker (1951 – 2003)

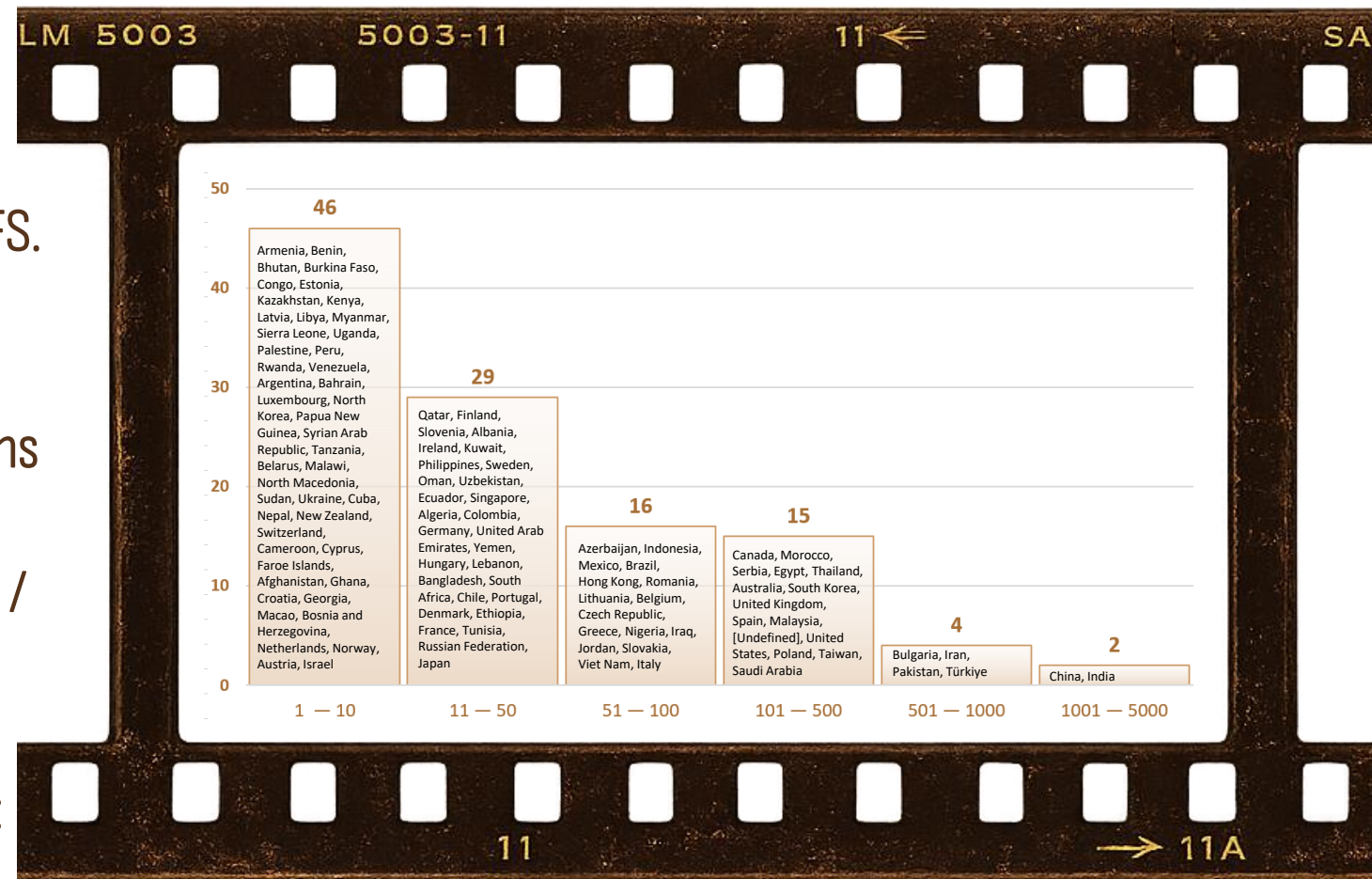


Intuitionistic Fuzzy Sets: 40-Years Legacy in Numbers

International Outreach

Researchers from more than 100 countries have published papers on IFS.

In Scopus, as of 31 Dec 2025, there are > 12 100 indexed publications with “intuitionistic fuzzy” in title / abstract / keywords, limited to article / conference paper / book chapter / conference review / book, and their distribution per country is given here:

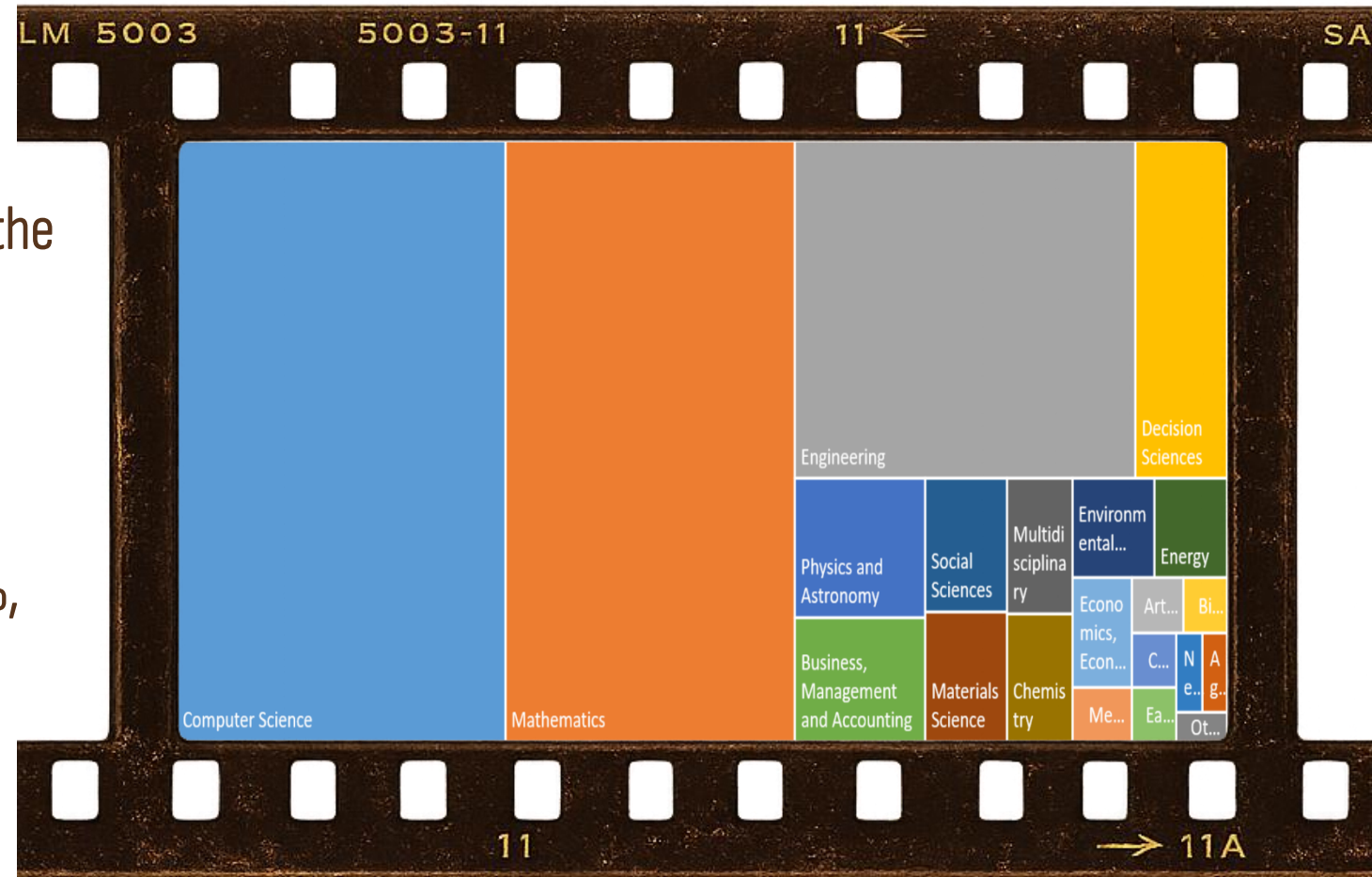


Intuitionistic Fuzzy Sets: 40-Years Legacy in Numbers

Thematic Diversity

Another perspective towards the impact of the 1986 paper is given by the diversity of the subject areas of the citing documents.

Computer Science, Mathematics and Engineering naturally lead with > 31%, > 27% and > 18% of all citations, but all major branches of science and technology are also represented with at least 100 papers each.

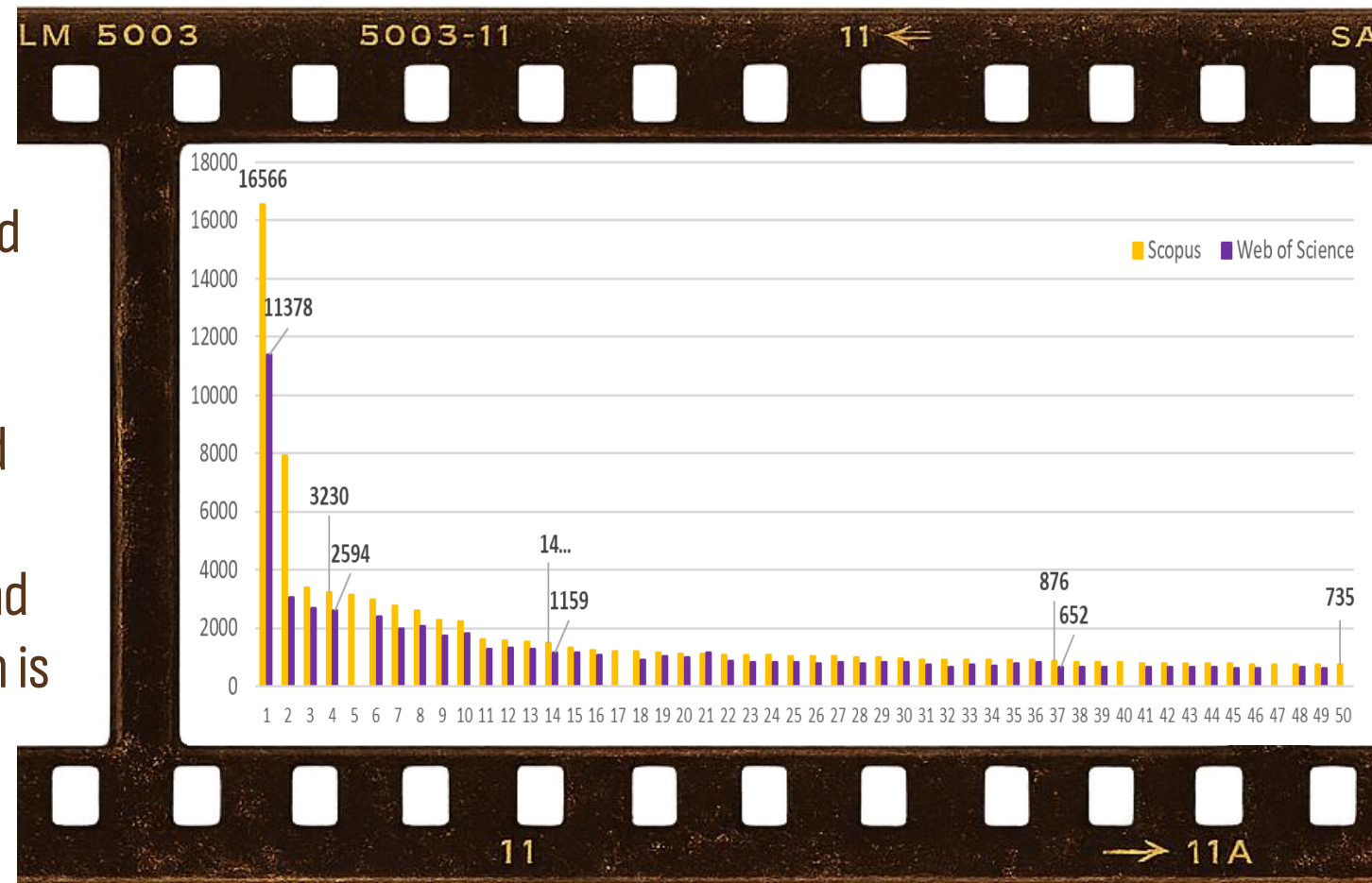


Intuitionistic Fuzzy Sets: 40-Years Legacy in Numbers

Impact on the Journal

This is also *the most cited publication* among nearly 10,000 papers published in the *"Fuzzy Sets and Systems"*.

With > 16 500 citations in Scopus and > 11 300 citations in Web of Science, it exceeds more than twice the second most cited paper in the Journal, which is the very first paper published in the *FS&S* Journal in its inaugural issue in 1978, authored by Lotfi Zadeh.

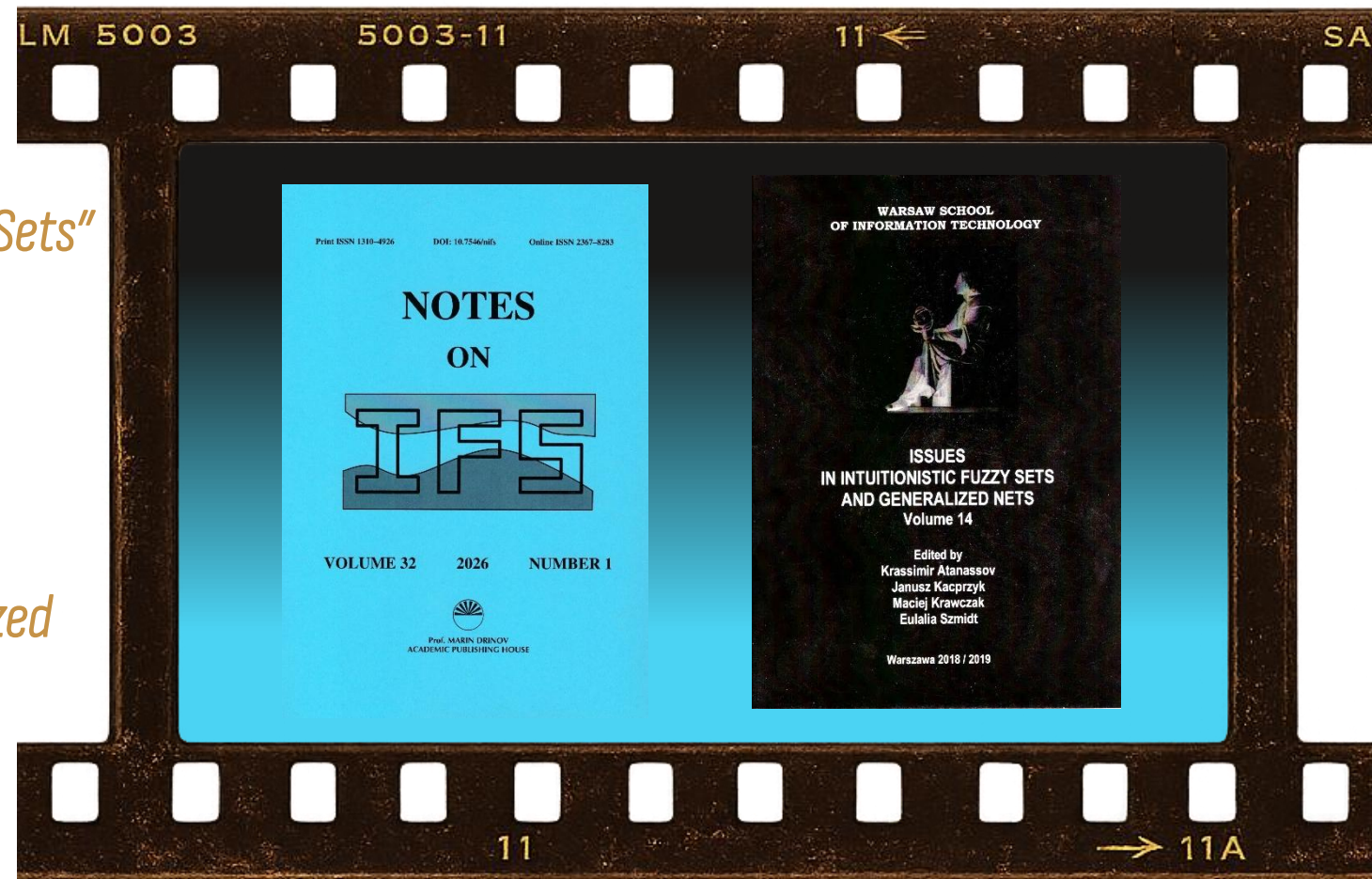


Intuitionistic Fuzzy Sets: Looking Ahead

Dedicated Journals

In Bulgaria in 1995, the international journal "*Notes on Intuitionistic Fuzzy Sets*" was founded by Atanassov, Kacprzyk and Bustince Sola, and is currently a Scopus-indexed Q3 journal.

In Poland, the book series "*Issues on Intuitionistic Fuzzy Sets and Generalized Nets*" used to be published annually from 2004 to 2019.



Intuitionistic Fuzzy Sets: Looking Ahead

Dedicated Conferences

International Conference on IFS, held in Bulgaria since 1997.

International Workshop on IFS and Generalized Nets, organized in Poland (Polish Academy of Sciences) since 2001.

Workshop on IFS, organized in Slovakia (Matej Bel University) since 2005.

IFSCOM conference, held in Türkiye (Mersin University) since 2014.



Intuitionistic Fuzzy Sets: Looking Ahead

Research Community

Collectively, these achievements and initiatives highlight a dynamic and globally connected research community whose ongoing efforts continue to shape and expand the field of IFSs.

Strengthening theoretical foundations, expanding application domains, and deeper integration with data-driven and artificial intelligence frameworks will remain key to sustaining their impact.



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Thank you
for your attention!

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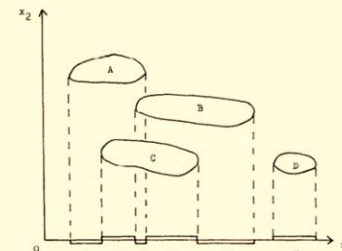


Fig. 1.

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