A Short Remark on a New Intuitionistic Fuzzy Implication

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In memory of my wife, Lilija C. Atanassova
(8 July 1956–6 August 2026).
She was a very good mathematician.

1. Introduction

The concept of an intuitionistic fuzzy implication was introduced in 1988. The subsequent development of the subject has produced many distinct intuitionistic fuzzy implications, reflecting different ways of combining the degree of membership, non-membership, and indeterminacy. Now, there are more than 200 intuitionistic fuzzy implications. Their definitions and some of properties are discussed in publications of H. Bustince and V. Mohedano; C. Cornelis, G. Deschrijver and E. Kerre, V. Tasseva and D. Peneva; L. Atanassova; P. Dworniczak; E. Szmidt and J. Kacprzyk and others.
A systematic study of the properties of intuitionistic fuzzy implications is developed in a series of five papers.
In the present paper, we introduce a new operation on intuitionistic fuzzy pairs and establish their properties. We also construct its dual coimplication.

For the purposes of this presentation, we use the notion of an intuitionistic fuzzy pair (IFP) . Let

$$\mathcal{P} = \{\langle a, b\rangle \in [0,1]^2 : a + b \leq 1\}.$$

The standard partial order on $\mathcal{P}$ is

$$\langle a, b\rangle \leq \langle c, d\rangle \quad \Longleftrightarrow \quad a \leq c \text{ and } b \geq d.$$

The distinguished pairs

$$\mathbf{O} = \langle 0, 1\rangle, \qquad \mathbf{U} = \langle 0, 0\rangle, \qquad \mathbf{E} = \langle 1, 0\rangle$$

represent falsity, complete uncertainty, and truth, respectively. An IFP $\langle p, q\rangle$ will be called an intuitionistic fuzzy tautology (IFT) when $p \geq q$; it is a tautology when it equals $\mathbf{E}$.

Throughout the presentation, let

$$x = \langle a, b \rangle, \qquad y = \langle c, d \rangle, \qquad z = \langle e, f \rangle$$

be arbitrary elements of $\mathcal{P}$.

2. The new intuitionistic fuzzy implication

The new intuitionistic fuzzy implication is defined by

$$x \Rightarrow_* y = \langle \max\{0, b-a, c-d\},\ \min\{a, d\}\rangle \tag{1}$$

Proposition

The definition of the new intuitionistic fuzzy implication is correct.

Proof. Put $P = \max\{0, b-a, c-d\}$ and $Q = \min\{a, d\}$.
Clearly, $P, Q \in [0, 1]$. It remains to prove $P + Q \leq 1$.
If $a \geq d$, then $Q = d$ and

$$P + Q = \max\{d, b-a+d, c\} \leq 1,$$

because $b - a + d \leq b \leq 1$. If $a < d$, then $Q = a$ and

$$P + Q = \max\{a, b, c-d+a\} \leq 1,$$

because $c - d + a < c \leq 1$. Thus $P + Q \leq 1$ in both cases, so $\langle P, Q\rangle \in \mathcal{P}$. $\square$

Proposition

The intuitionistic fuzzy implication $\Rightarrow_*$ is decreasing in its first argument and increasing in its second argument. More precisely,

$$x \leq y \Longrightarrow x \Rightarrow_* z \geq y \Rightarrow_* z \quad \textit{and} \quad x \leq y \Longrightarrow z \Rightarrow_* x \leq z \Rightarrow_* y.$$

Moreover,

$$\mathbf{O} \Rightarrow_* x = \mathbf{E}, \qquad x \Rightarrow_* \mathbf{E} = \mathbf{E}, \qquad \mathbf{E} \Rightarrow_* \mathbf{O} = \mathbf{O}.$$

Proof. Suppose that $x \leq y$, so $a \leq c$ and $b \geq d$. Hence $b - a \geq d - c$, and therefore

$$\max\{0, b-a, e-f\} \geq \max\{0, d-c, e-f\}, \qquad \min\{a, f\} \leq \min\{c, f\}.$$

These are precisely the two component inequalities required for $x \Rightarrow_* z \geq y \Rightarrow_* z$.

Similarly, from $a - b \leq c - d$, it follows that

$$\max\{0, f-e, a-b\} \leq \max\{0, f-e, c-d\}, \qquad \min\{e, b\} \geq \min\{e, d\}.$$

Thus $z \Rightarrow_* x \leq z \Rightarrow_* y$. The three boundary identities follow directly from (1). $\square$

3. The generated negation

The negation generated by $\Rightarrow_*$ is defined, as usual, by $\neg_* x = x \Rightarrow_* \mathbf{O}$. For every $x = \langle a, b\rangle \in \mathcal{P}$,

$$\neg_*\langle a, b\rangle = \langle \max\{0, b-a\}, a\rangle. \tag{2}$$

Proposition

The intuitionistic fuzzy negation $\neg_*$ reverses the order on $\mathcal{P}$ and satisfies

$$\neg_*\mathbf{O} = \mathbf{E}, \qquad \neg_*\mathbf{E} = \mathbf{O}, \qquad \neg_*\mathbf{U} = \mathbf{U}.$$

In general, however, it is not involutive and hence is not a strong intuitionistic fuzzy negation.

Proof. Formula (2) follows immediately from (1). Order reversal is a special case of the decrease of $\Rightarrow_*$ in its first argument. If $p = \max\{0, b-a\}$, then

$$\neg_*\neg_*\langle a, b\rangle = \langle \max\{0, a-p\}, p\rangle = \langle \max\{0, \min\{a, 2a-b\}\}, \max\{0, b-a\}\rangle\,.$$

For example,

$$\neg_*\neg_*\langle \tfrac{1}{3}, \tfrac{1}{3}\rangle = \langle \tfrac{1}{3}, 0\rangle \neq \langle \tfrac{1}{3}, \tfrac{1}{3}\rangle.$$

Thus $\neg_*$ is not involutive. $\square$

Obviously, the new negation is not classical one. It is different than the remaining intuitionistic fuzzy negations.

4. Axiomatic properties

Now, we consider the following axioms for an implication $I : \mathcal{P}^2 \to \mathcal{P}$, adapted from the standard fuzzy implication axioms in the book of G. Klir and B. Yuan from 1995 and their intuitionistic fuzzy forms:

$$\begin{array}{ll}
(A1) & x \leq y \Longrightarrow I(x,z) \geq I(y,z), \\
(A2) & x \leq y \Longrightarrow I(z,x) \leq I(z,y), \\
(A3) & I(\mathbf{O}, y) = \mathbf{E}, \\
(A4) & I(\mathbf{E}, y) = y, \\
(A5) & I(x,x) = \mathbf{E}, \\
(A6) & I(x, I(y,z)) = I(y, I(x,z)), \\
(A7) & I(x,y) = \mathbf{E} \quad \Longleftrightarrow \quad x \leq y, \\
(A8) & I(x,y) = I(N(y), N(x)), \\
(A9) & I \text{ is continuous.}
\end{array}$$

Here N in $(A8)$ is the negation generated by I.

The following weakened or modified forms are also considered:

$$
\begin{array}{ll}
(A3^*) & I(\mathbf{O}, y) \text{ is an IFT,} \\
(A4^*) & I(\mathbf{E}, y) \leq y, \\
(A5^*) & I(x, x) \text{ is an IFT,} \\
(A7^*) & x \leq y \Longrightarrow I(x, y) = \mathbf{E}, \\
(A8^*) & I(x, y) = N^2(I(N(y), N(x))).
\end{array}
$$

Here, we additionally introduce the following restricted forms of the axioms:

$(A5^o)$ $\quad x = \langle a, 0\rangle$ or $x = \langle 0, b\rangle \Longrightarrow I(x, x)$ is an IFT,

$(A6^o)$ $\quad x = \mathbf{O}$ or $y = \mathbf{O}$ or $z = \langle e, 0\rangle \Longrightarrow I(x, I(y, z)) = I(y, I(x, z))$,

$(A7^o)$ $\quad I(x, y) = \mathbf{E} \Longrightarrow x \le y$,

$(A8^o)$ $\quad I(x, y) = I(\neg_1(y), \neg_1(x))$,

where

$$\neg_1\langle a, b\rangle = \langle b, a\rangle \tag{3}$$

is the standard strong intuitionistic fuzzy negation.

Theorem

The new intuitionistic fuzzy implication $\Rightarrow_*$ satisfies

$$A1,\ A2,\ A3,\ A3^*,\ A4^*,\ A5^o,\ A6^o,\ A7^o,\ A8^o,\ \textit{and}\ A9.$$

It does not satisfy, in general,

$$A4,\ A5,\ A5^*,\ A6,\ A7,\ A7^*,\ A8,\ \textit{or}\ A8^*.$$

Proof.

Axioms $A1$, $A2$, $A3$, and hence $A3^*$, follow from Proposition 2.
For $A4^*$,

$$\mathbf{E} \Rightarrow_* \langle a, b\rangle = \langle \max\{0, a-b\}, b\rangle \leq \langle a, b\rangle,$$

because $\max\{0, a-b\} \leq a$. Equality does not hold in general; for example,

$$\mathbf{E} \Rightarrow_* \langle \tfrac{2}{5}, \tfrac{1}{10}\rangle = \langle \tfrac{3}{10}, \tfrac{1}{10}\rangle.$$

Thus $A4$ fails.
Furthermore,

$$x \Rightarrow_* x = \langle |a-b|, \min\{a, b\}\rangle. \tag{4}$$

If $a = 0$ or $b = 0$, this pair has the form $\langle \max\{a, b\}, 0\rangle$ and is an IFT. Hence $A5^o$ holds. On the other hand,

$$\langle \tfrac{1}{3}, \tfrac{1}{3}\rangle \Rightarrow_* \langle \tfrac{1}{3}, \tfrac{1}{3}\rangle = \langle 0, \tfrac{1}{3}\rangle,$$

which is neither an IFT nor $\mathbf{E}$. Consequently, both $A5$ and $A5^*$ fail. If $z = \langle e, 0\rangle$, direct substitution gives

$$\begin{aligned}
&x \Rightarrow_* (y \Rightarrow_* z) = y \Rightarrow_* (x \Rightarrow_* z)\\
&= \langle \max\{0, b - a, d - c, e\}, 0\rangle\\
&= \langle \max\{b - a, d - c, e\}, 0\rangle.
\end{aligned}$$

If $x = \mathbf{O}$, both sides of $A6$ equal $\mathbf{E}$, and the case $y = \mathbf{O}$ is symmetric. Thus $A6^o$ holds. The unrestricted exchange principle $A6$ fails. For instance, with

$$x = \mathbf{U}, \qquad y = \langle \tfrac{1}{4}, 0 \rangle, \qquad z = \langle \tfrac{1}{2}, \tfrac{1}{4} \rangle,$$

one obtains

$$x \Rightarrow_* (y \Rightarrow_* z) = \langle 0, 0 \rangle, \qquad y \Rightarrow_* (x \Rightarrow_* z) = \langle \tfrac{1}{4}, 0 \rangle.$$

Suppose that $x \Rightarrow_* y = \mathbf{E}$. Then

$$\max\{0, b-a, c-d\} = 1, \qquad \min\{a, d\} = 0.$$

Since $b - a \leq 1$ and $c - d \leq 1$, either $b - a = 1$, which implies $x = \mathbf{O}$, or $c - d = 1$, which implies $y = \mathbf{E}$. In either case $x \leq y$; hence $A7^o$ holds. The converse fails because $\mathbf{U} \leq \mathbf{U}$, whereas $\mathbf{U} \Rightarrow_* \mathbf{U} = \mathbf{U} \neq \mathbf{E}$. Thus $A7$ and $A7^*$ fail.
Using (3), we obtain

$$\begin{aligned}\neg_1(y) \Rightarrow_* \neg_1(x) &= \langle d, c\rangle \Rightarrow_* \langle b, a\rangle \\ &= \langle \max\{0, c - d, b - a\}, \min\{d, a\}\rangle \\ &= x \Rightarrow_* y,\end{aligned}$$

so $A8^o$ holds. Neither $A8$ nor $A8^*$ holds for the generated negation $\neg_*$. Indeed, if $x = \mathbf{U}$ and $y = \langle \frac{1}{3}, \frac{1}{3}\rangle$, then

$$x \Rightarrow_* y = \mathbf{U}, \qquad \neg_*(y) \Rightarrow_* \neg_*(x) = \langle \tfrac{1}{3}, 0\rangle,$$

and applying $\neg_*^2$ to the latter pair leaves it unchanged.
Finally, $A9$ follows because the maximum and minimum of finitely many continuous functions are continuous. $\square$

5. Comparison with related implications

The following three implications are structurally close to $\Rightarrow_*$

$$
\begin{aligned}
x \rightarrow_1 y &= \langle \max\{b, \min\{a, c\}\}, \min\{a, d\}\rangle, \\
x \rightarrow_4 y &= \langle \max\{b, c\}, \min\{a, d\}\rangle, \\
x \rightarrow_{61} y &= \langle \max\{c, \min\{b, d\}\}, \min\{a, d\}\rangle.
\end{aligned}
$$

We can see directly that

$$x \Rightarrow_* y \leq x \rightarrow_4 y,$$

but the other two operations are incomparable to $\Rightarrow_*$.

6. The dual coimplication

Let $\neg_1$ be the standard negation in (3). Following the usual duality principle for implications and coimplications, define

$$C_*(x, y) = \neg_1\big(\neg_1(x) \Rightarrow_* \neg_1(y)\big). \tag{5}$$

Proposition

The operation dual to $\Rightarrow_*$ is

$$C_*\big(\langle a, b\rangle, \langle c, d\rangle\big) = \langle \min\{b, c\}, \max\{0, a - b, d - c\}\rangle. \tag{6}$$

It is decreasing in its first argument, increasing in its second argument, and satisfies

$$C_*(x, \mathbf{O}) = \mathbf{O}, \qquad C_*(\mathbf{E}, x) = \mathbf{O}, \qquad C_*(\mathbf{O}, \mathbf{E}) = \mathbf{E}.$$

On Boolean pairs it reduces to $\neg x \wedge y$.

Proof.

Since $\neg_1\langle a, b\rangle = \langle b, a\rangle$, formula (5) gives

$$\begin{aligned} C_*(x, y) &= \neg_1\left(\langle b, a\rangle \Rightarrow_* \langle d, c\rangle\right) \\ &= \neg_1\langle \max\{0, a-b, d-c\}, \min\{b, c\}\rangle \\ &= \neg_1(y \Rightarrow_* x). \end{aligned}$$

Thus the coimplication is the standard negation of the implication with its arguments reversed. $\square$

7. Conclusion

The new intuitionistic fuzzy implication, like many other intuitionistic fuzzy implications, does not satisfy some of the axioms of G. Klir and B. Yuan, but they were initially introduced for fuzzy, not for intuitionistic fuzzy implications. The present study clearly shows that a new axiomatic system related precisely to intuitionistic fuzzy implications must be introduced, which is currently an open problem. The relationship of the present implication with the other intuitionistic fuzzy implications will be discussed in the near future.

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Thank you for attention!