Living on the perimeter: A new directional operator on intuitionistic fuzzy sets

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Introduction

Over the past almost 40 years, Intuitionistic Fuzzy Interpretational Triangle (IFIT) has established itself as a useful tool not only for illustrating the degrees of membership (validity, correctness, etc.) and of non-membership (non-validity, incorrectness, etc.), but also as a generator for the emergence of new ideas for defining operations and operators on intuitionistic fuzzy objects.

The Intuitionistic Fuzzy Pair (IFP) is an object with the form $\langle a, b \rangle$, where $a, b \in [0, 1]$ and $a + b \leq 1$, that is used as an evaluation of some object or process.

The most important geometrical interpretation of the IFPs using IFIT is shown on Figs. 1.-25

Let us have two IFPs $x = \langle a, b \rangle$ and $y = \langle c, d \rangle$. Now, we define the relations

$$
\begin{aligned}
x < y \quad &\text{iff} \quad a < c \text{ and } b > d \\
x \leq y \quad &\text{iff} \quad a \leq c \text{ and } b \geq d \\
x > y \quad &\text{iff} \quad a > c \text{ and } b < d \\
x \geq y \quad &\text{iff} \quad a \geq c \text{ and } b \leq d \\
x = y \quad &\text{iff} \quad a = c \text{ and } b = d
\end{aligned}
$$

Operations from different types are defined over two IFPs, e.g.,

$$x \wedge y = \langle \min(a, c), \max(b, d) \rangle,$$

$$x \vee y = \langle \max(a, c), \min(b, d) \rangle,$$

$$x + y = \langle a + c - ac, bd \rangle,$$

$$x.y = \langle ac, b + d - bd \rangle,$$

$$x@y = \langle \tfrac{a+c}{2}, \tfrac{b+d}{2} \rangle,$$

and others.

In this study, we will introduce a new operation between two IFPs, which, as a result of directed extrapolation of these IFPs, juxtaposes an IFP over the IFIT boundary.

Main results

Initially, we will construct 16 IFITs corresponding to the different possible positions of two (different from each other) IFPs that do not lie on the side of the IFIT, and an operation "*" that matches these IFPs with the IFP that lies on one side of the IFIT. Then, we will show 6 cases in which both IFPs lie on one side of the IFIT, and finally, we will show 3 cases in which the first IFP coincides with one of the vertices of the IFIT and the other IFP is inside the IFIT.
Let

$$x * y = z = \langle e, f \rangle.$$

For each IFIT we will indicate the coordinates of the IFP z.

The essence of our constructions is that if x is the first IFP and y is the second IFP, we extend the segment from x through y until it reaches a side of the IFIT. The result is a new IFP on the corresponding side. The order of the IFPs (x, y) determines the direction, so there is no need for a third argument to the operation. Reversing the order of the arguments (y, x) selects the other end of the same segment.

For each configuration we will show the relationships between a, b, c, d.

Let us formulate the following four conditions that the constants a, b, c, d must satisfy in the different cases:

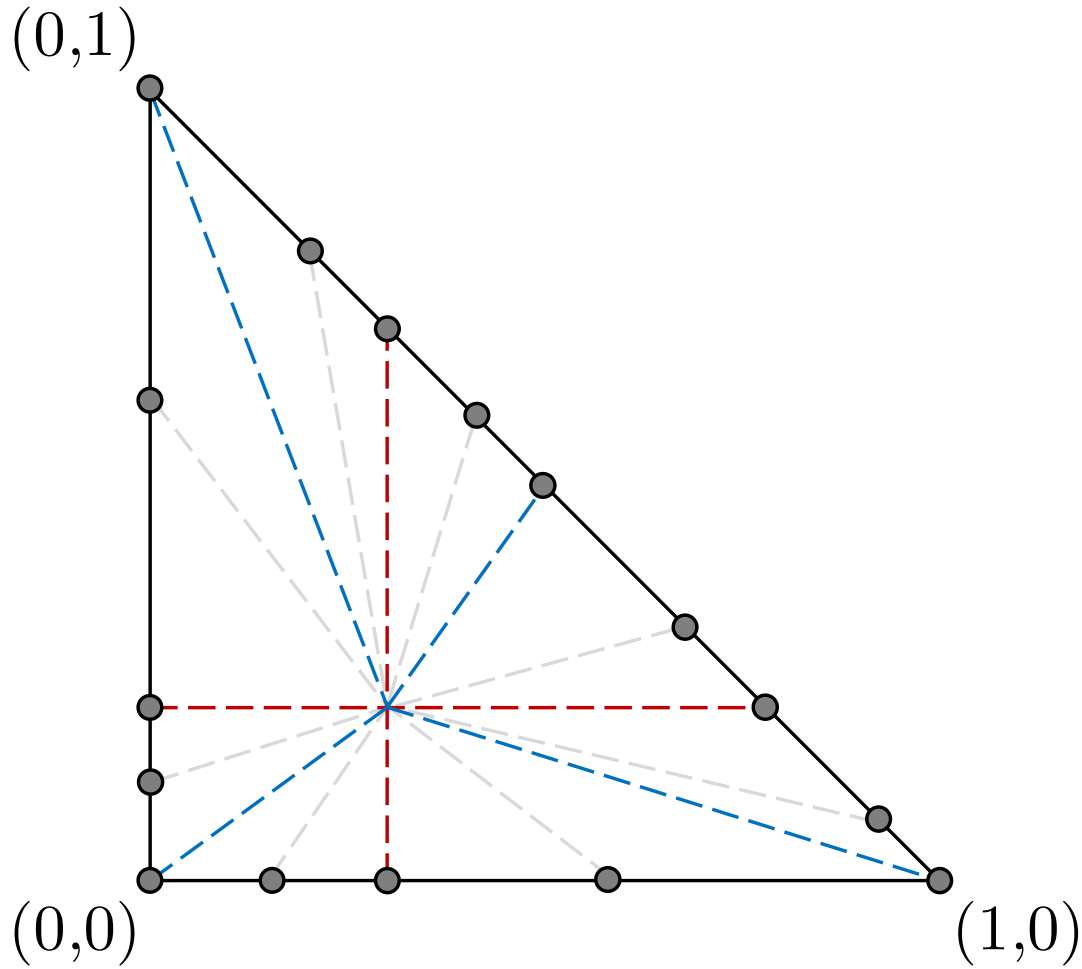
$$c(1-b) > a(1-d) \tag{1}$$

$$d(1-a) > b(1-c) \tag{2}$$

$$d(1-a) \leq b(1-c) \tag{3}$$

$$c(1-b) \leq a(1-d) \tag{4}$$

In the research, we will study the following cases.



All directions

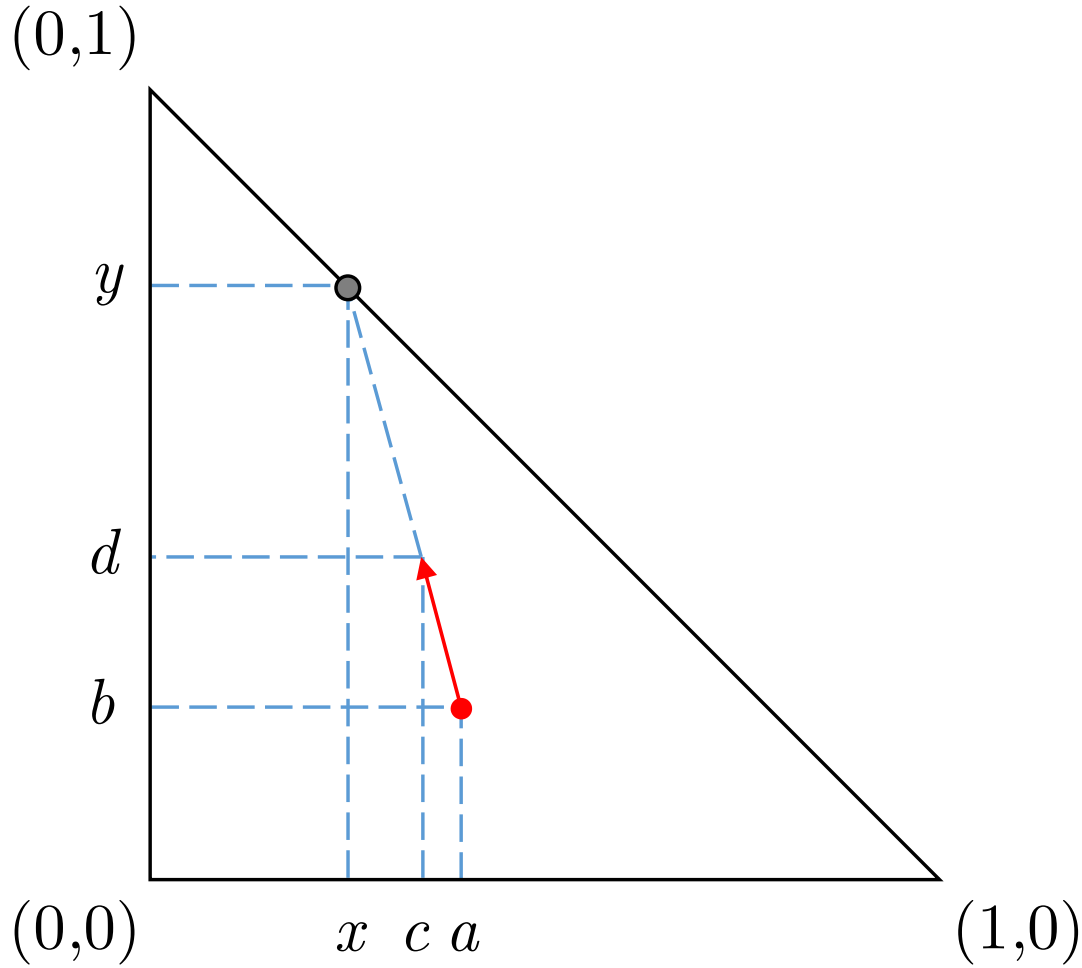


Fig. 1: $a > c, b < d, a + b < c + d$, (1), (2) and $z = \left\langle \frac{c-a+ad-bc}{c+d-a-b}, \frac{d-b-ad+bc}{c+d-a-b} \right\rangle$

Theorem 1. The definition of operation "*" is correct.

Proof: First, we see that

$$c - a + ad - bc = c(1 - b) - a(1 - d) > 0.$$

Second, we check that

$$\begin{aligned} c + d - a - b - c + a - ad + bc &= d - b - ad + bc \\ &= d - ad - b(1 - c) \\ &> d - ad - (c + d - a)(1 - c) \\ &= d - ad - c - d + a + c^2 + cd - ac \\ &= a(1 - c - d) - c(1 - c - d) \\ &= (a - c)(1 - c - d) > 0. \end{aligned}$$

Therefore, $x \in [0, 1]$. Hence, $y = 1 - x \in [0, 1]$ because the condition $x + y = 1$.

The check of the remaining formulas for x and y are similar. □

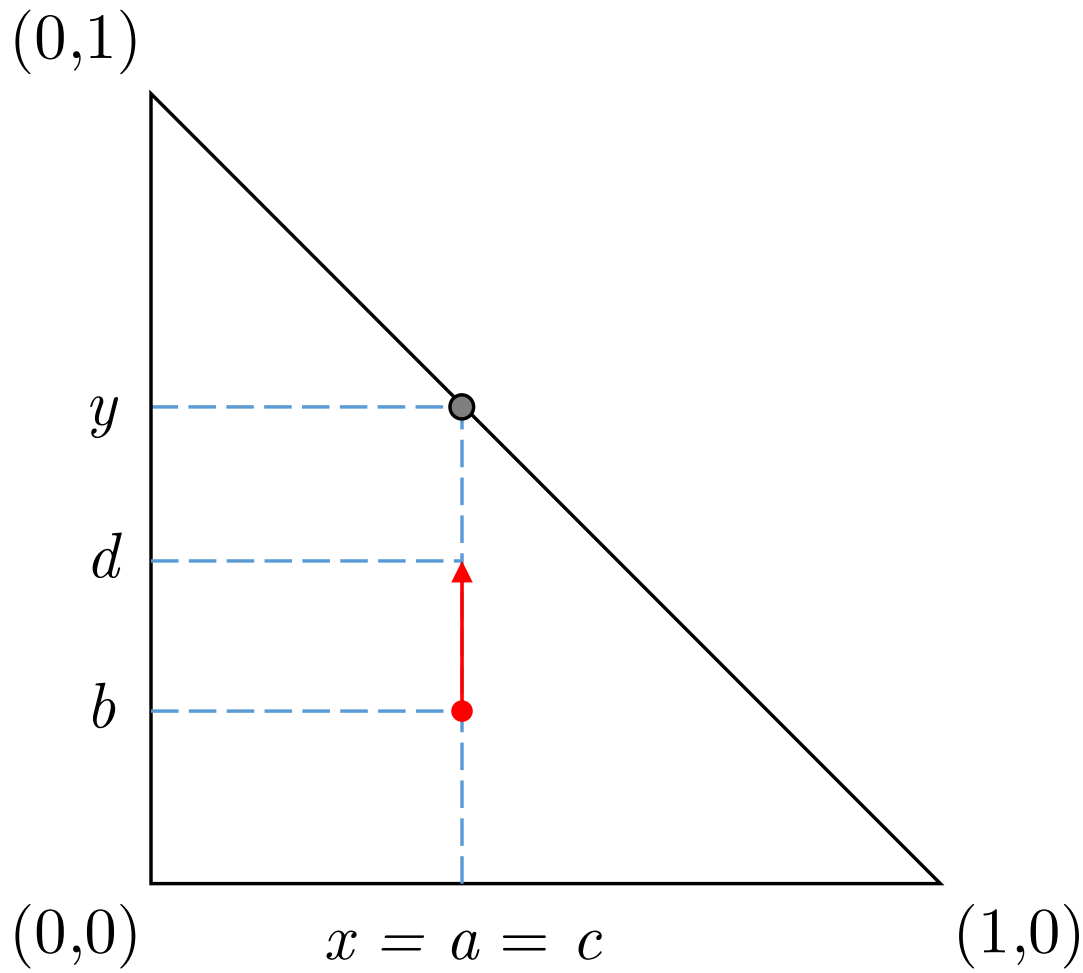


Fig. 2: $a = c, b < d$, (1), (2) and $z = \langle a, 1 - a \rangle$

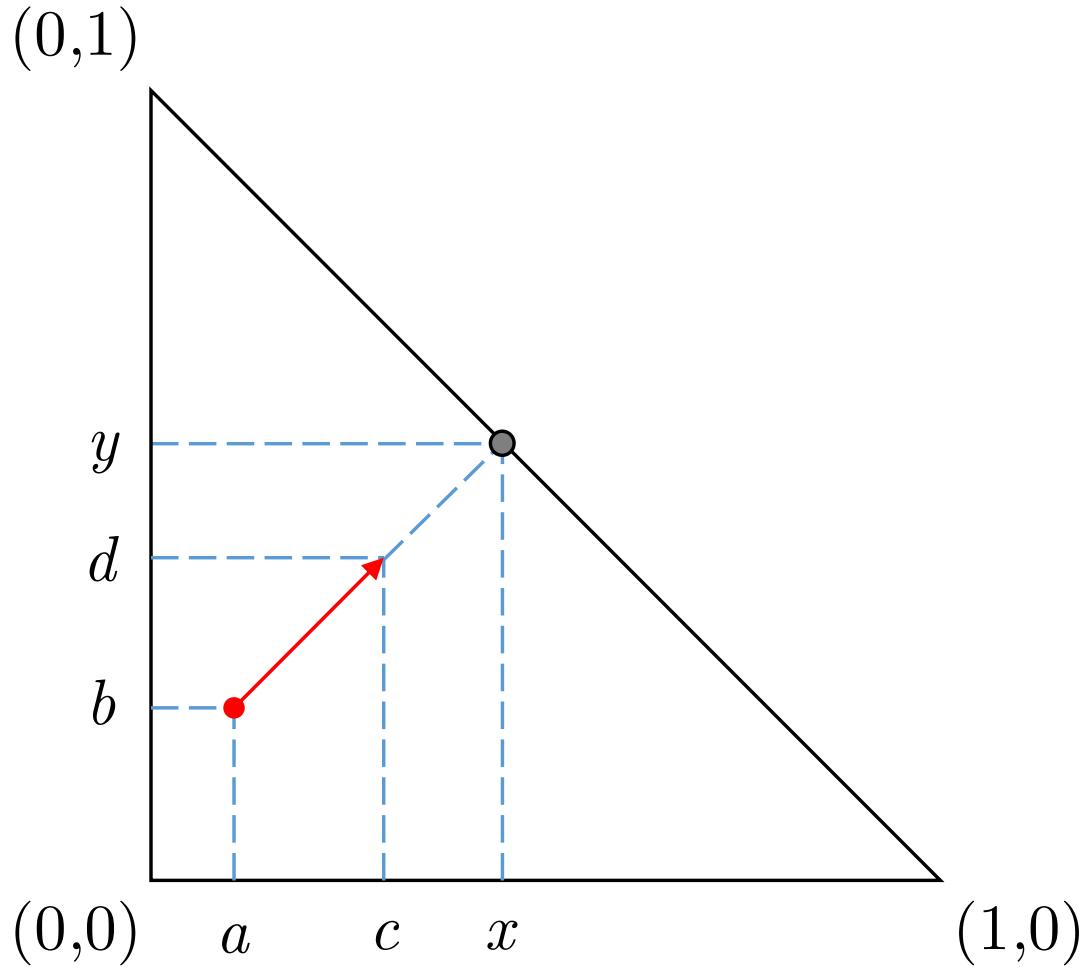


Fig. 3: $a < c, b < d$, (1), (2) and $z = \left\langle \frac{c-a+ad-bc}{c+d-a-b}, \frac{d-b-ad+bc}{c+d-a-b} \right\rangle$

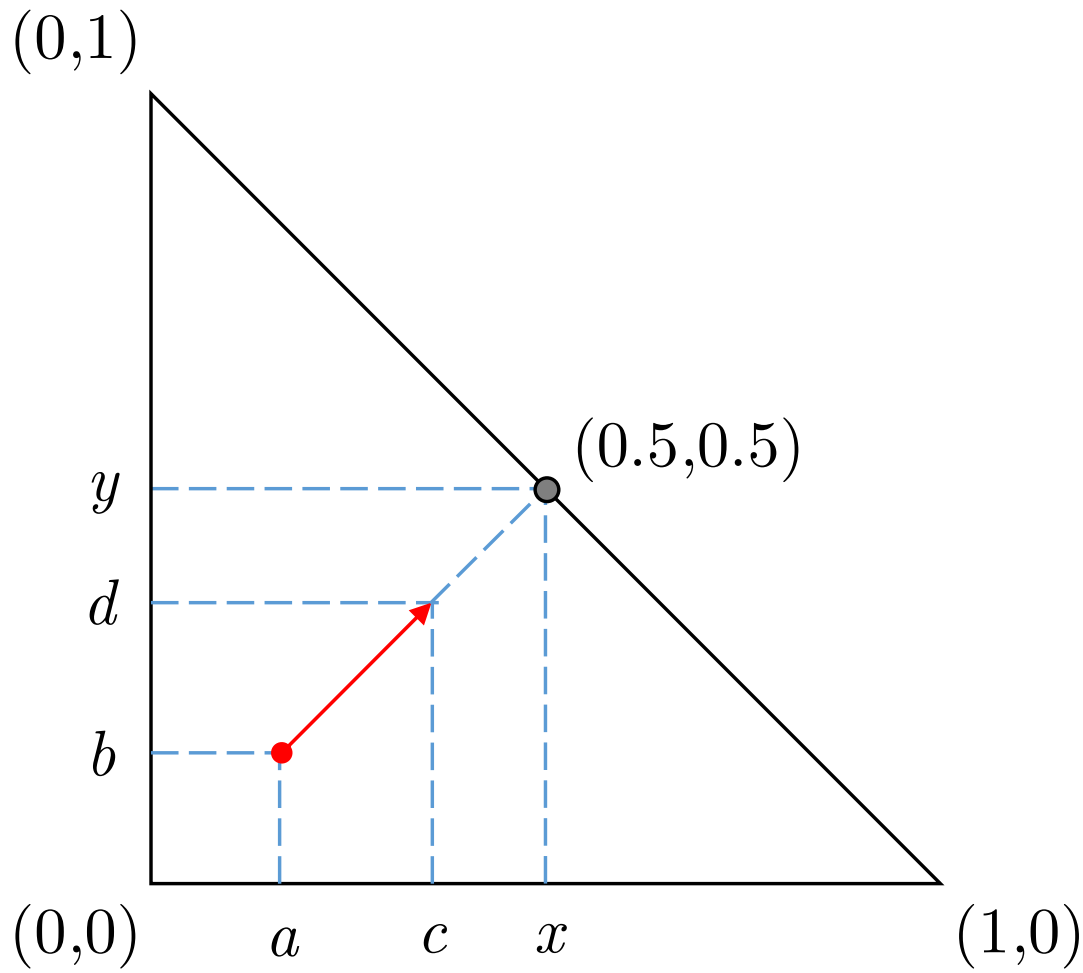


Fig. 4: $a < c, b < d, a = b, c = d$, (1), (2) and $z = \left\langle \frac{1}{2}, \frac{1}{2} \right\rangle$

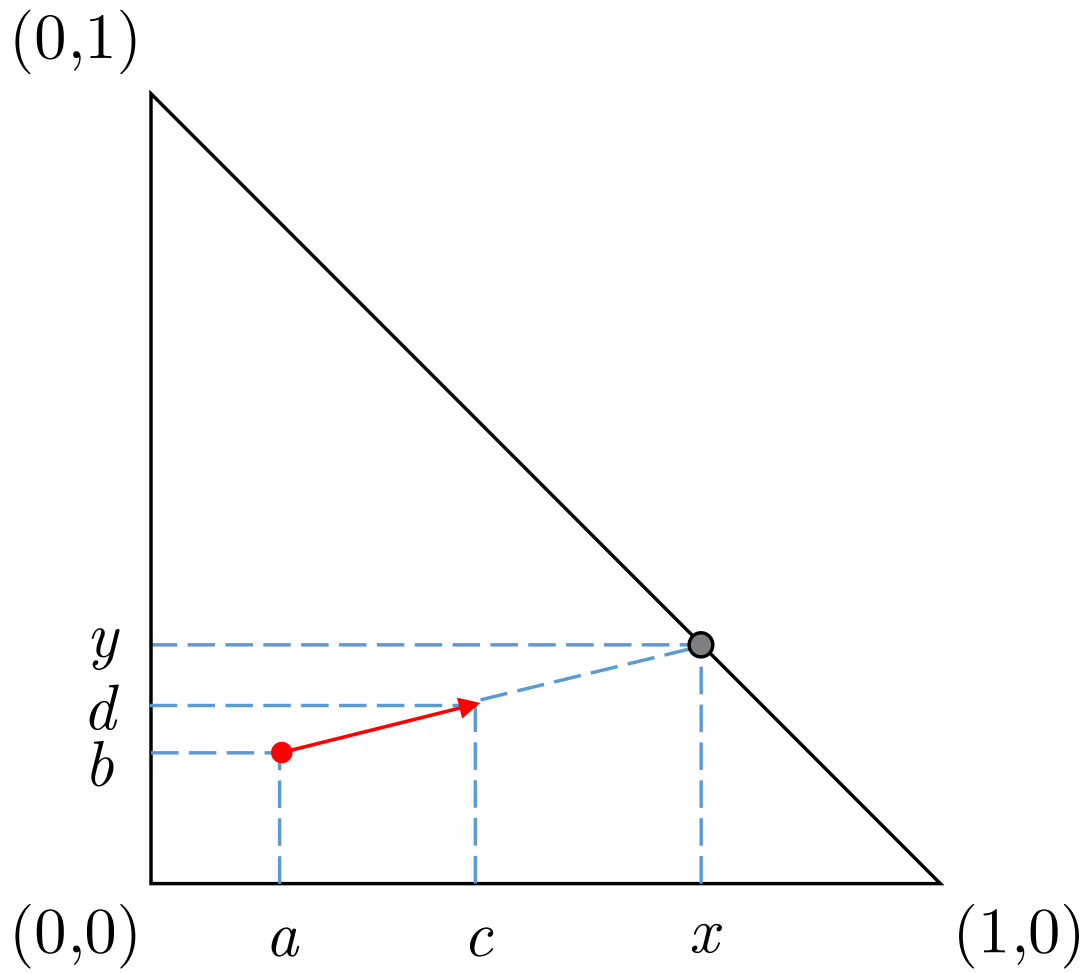


Fig. 5: $a < c, b < d$, (1), (2) and $z = \langle 1 - b, b \rangle$

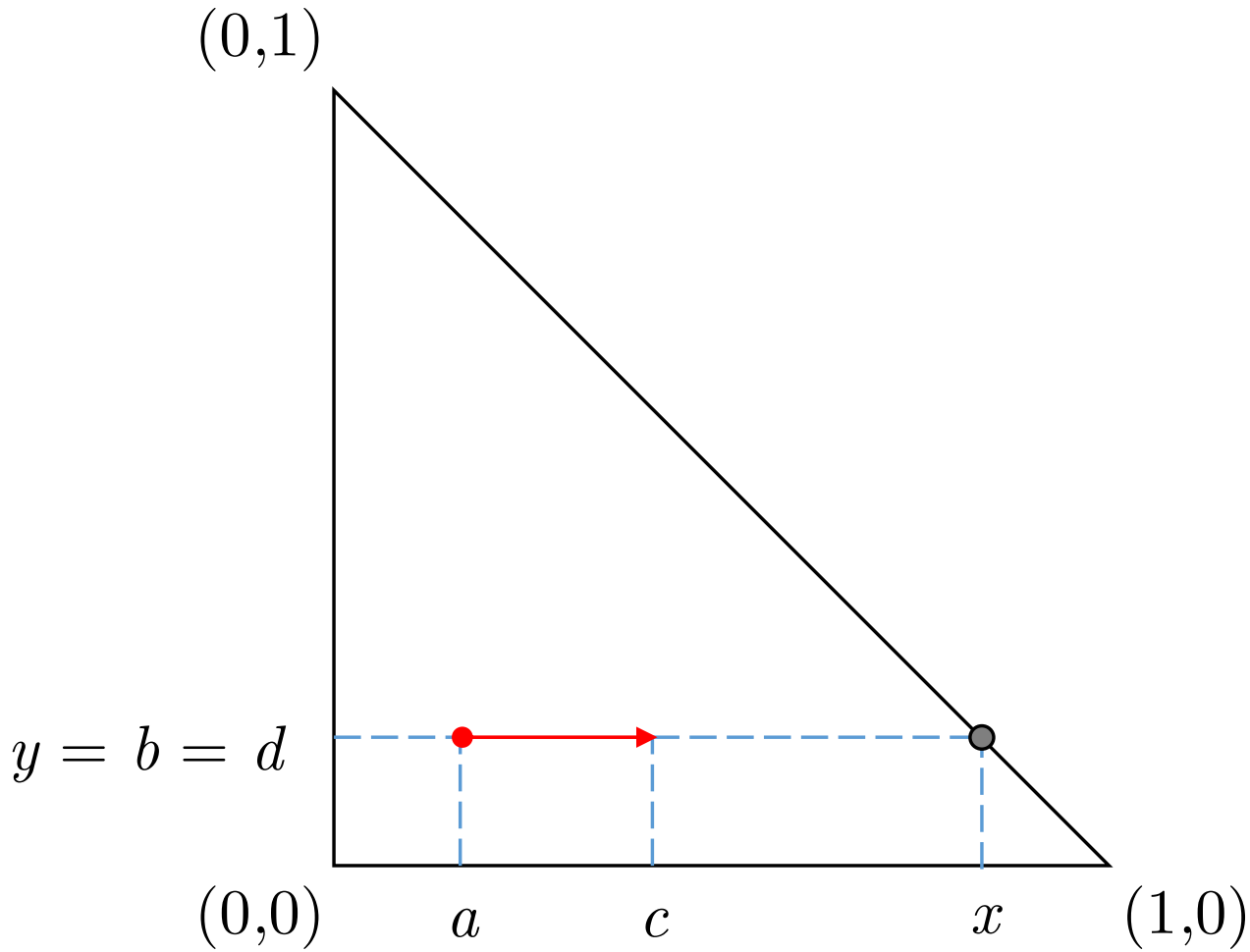


Fig. 6: $a < c, b = d$, (1), (2) and $z = \left\langle \frac{c-a+ad-bc}{c+d-a-b}, \frac{d-b-ad+bc}{c+d-a-b} \right\rangle$

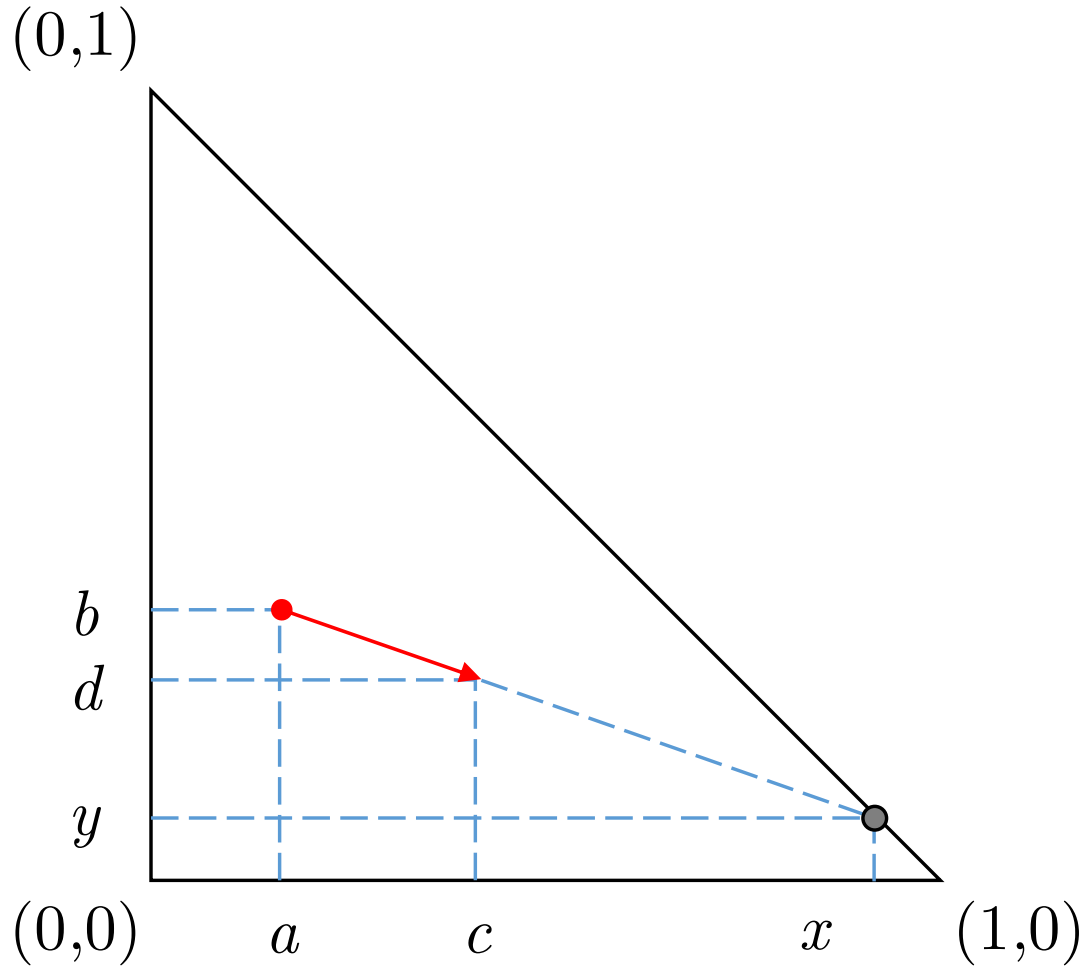


Fig. 7: $a < c, b < d$, (1), (2) and $z = \left\langle \frac{c-a+ad-bc}{c+d-a-b}, \frac{d-b-ad+bc}{c+d-a-b} \right\rangle$

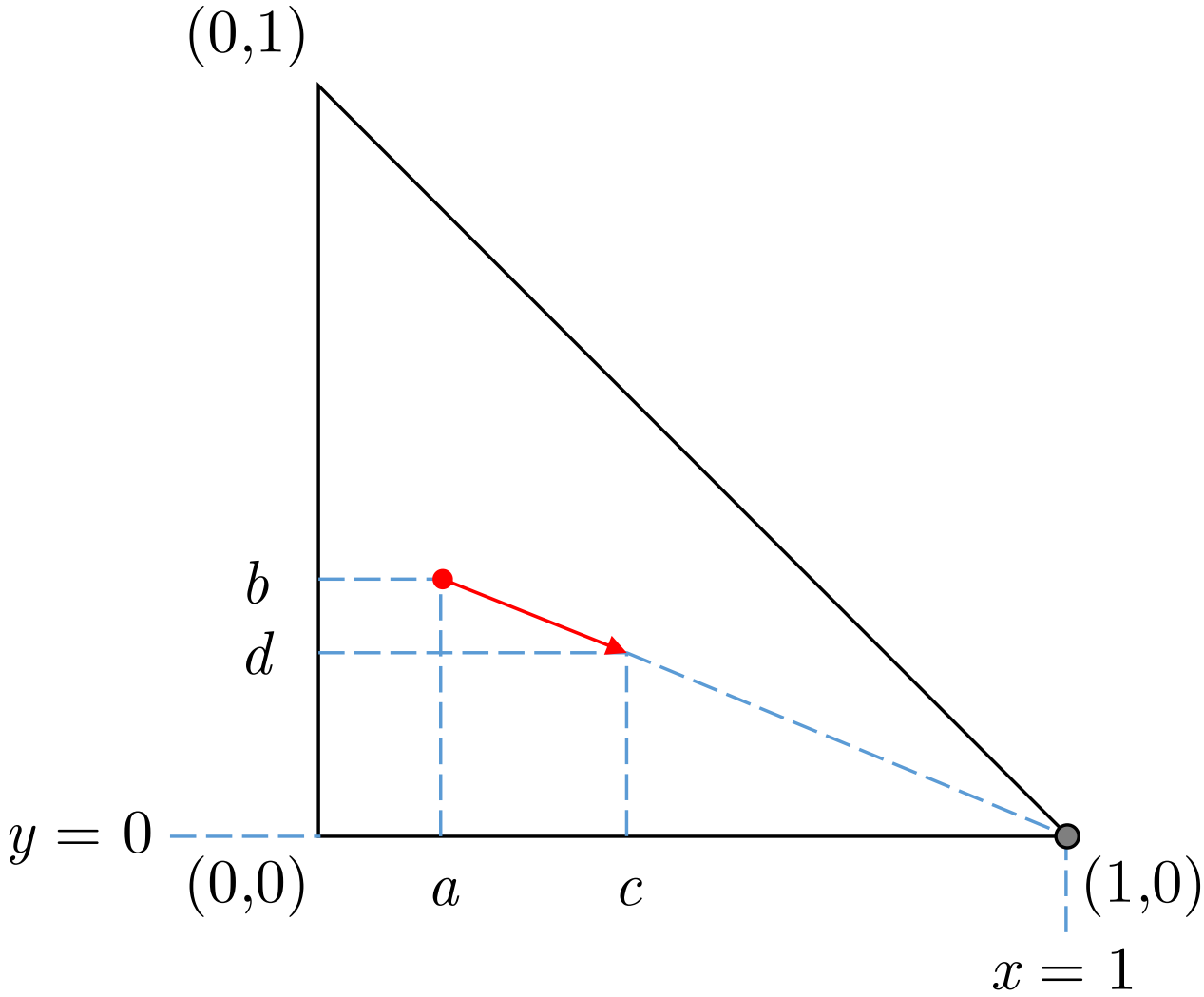


Fig. 8: $a < c, b > d, \frac{1-a}{1-c} = \frac{b}{d}$, (2) and $z = \langle 1, 0 \rangle$

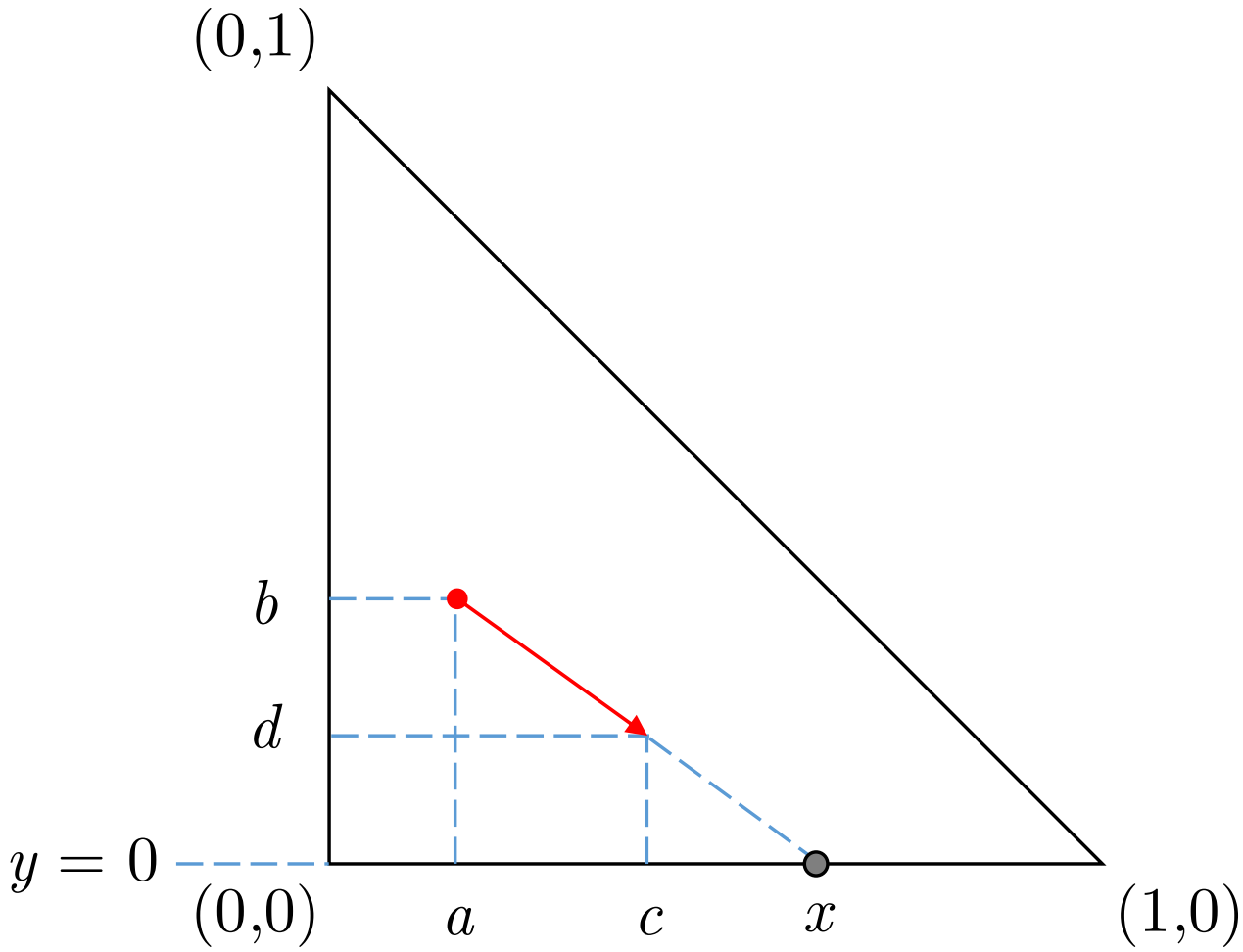


Fig. 9: $a < c, b > d$, $\frac{1-a}{1-c} = \frac{b}{d}$, (1), (3) and $z = \left\langle \frac{bc-ad}{b-d}, 0 \right\rangle$

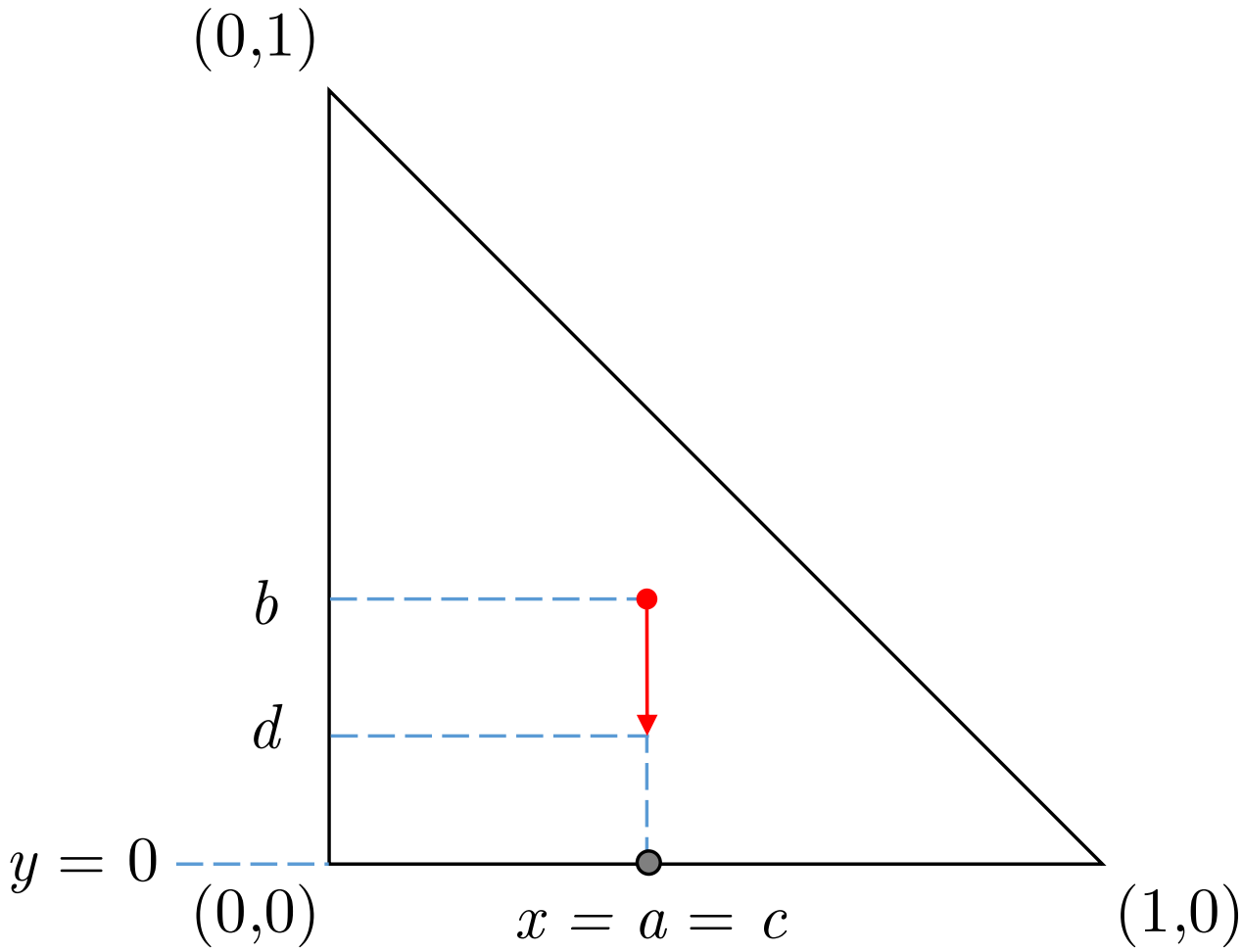


Fig. 10: $a = c, b > d$, (1), (3) and $z = \langle a, 0 \rangle$

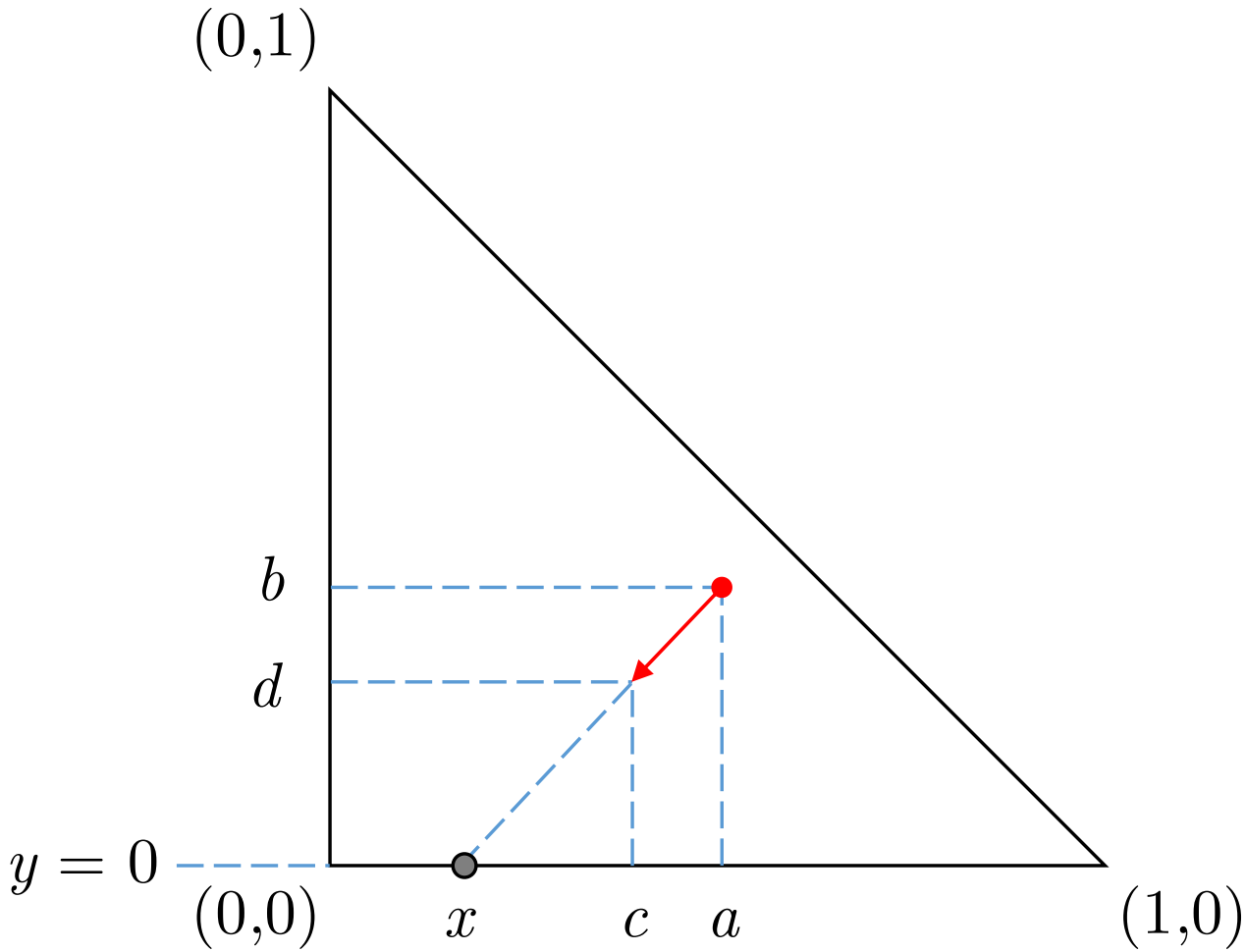


Fig. 11: $a > c, b > d, c > d$, (1), (3) and $z = \left\langle \frac{bc-ad}{b-d}, 0 \right\rangle$

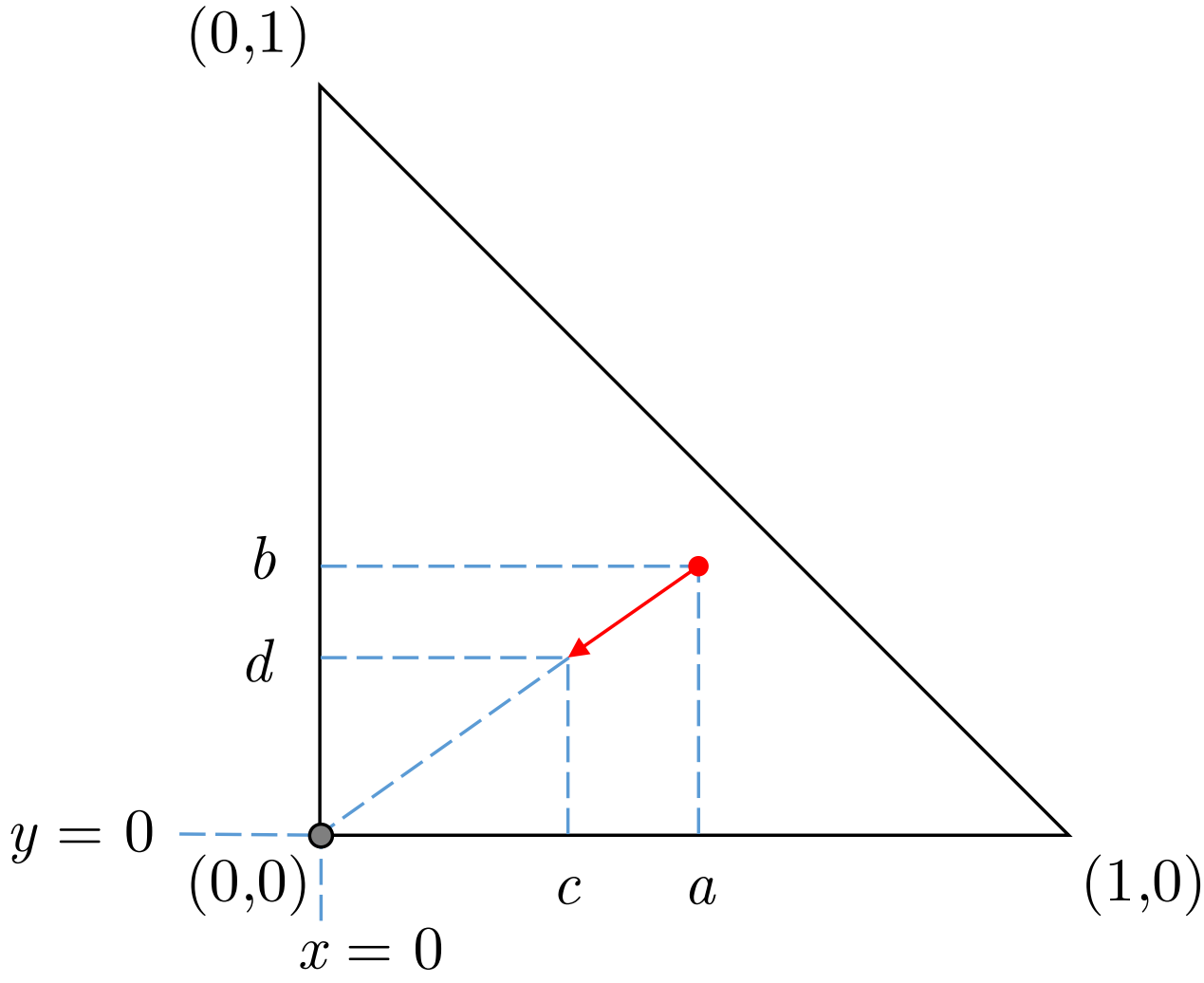


Fig. 12: $a > c, b > d$, $\frac{a}{c} = \frac{b}{d}$ and $z = \langle 0, 0 \rangle$

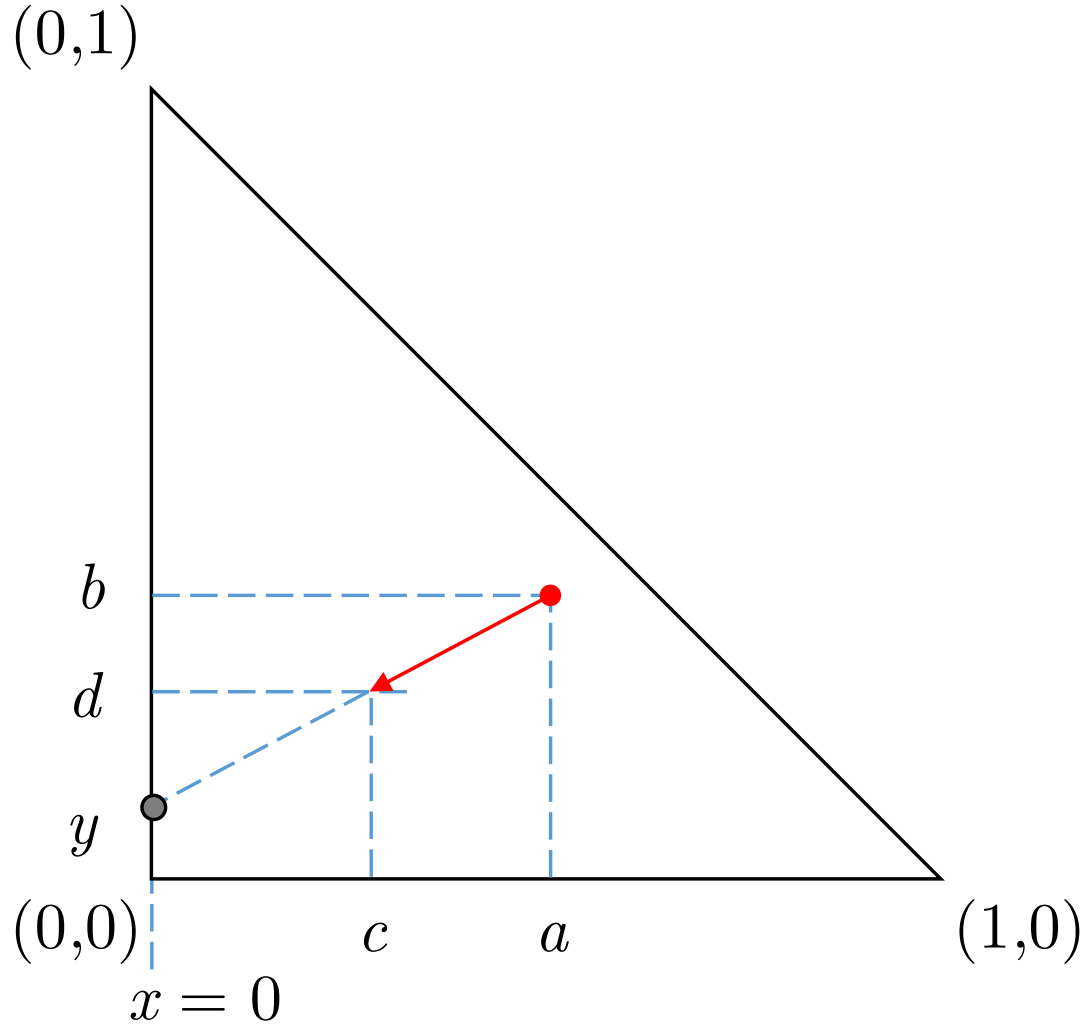


Fig. 13: $a > c, b > d$, (2), (4) and $z = \left\langle 0, \frac{ad-bc}{a-c} \right\rangle$

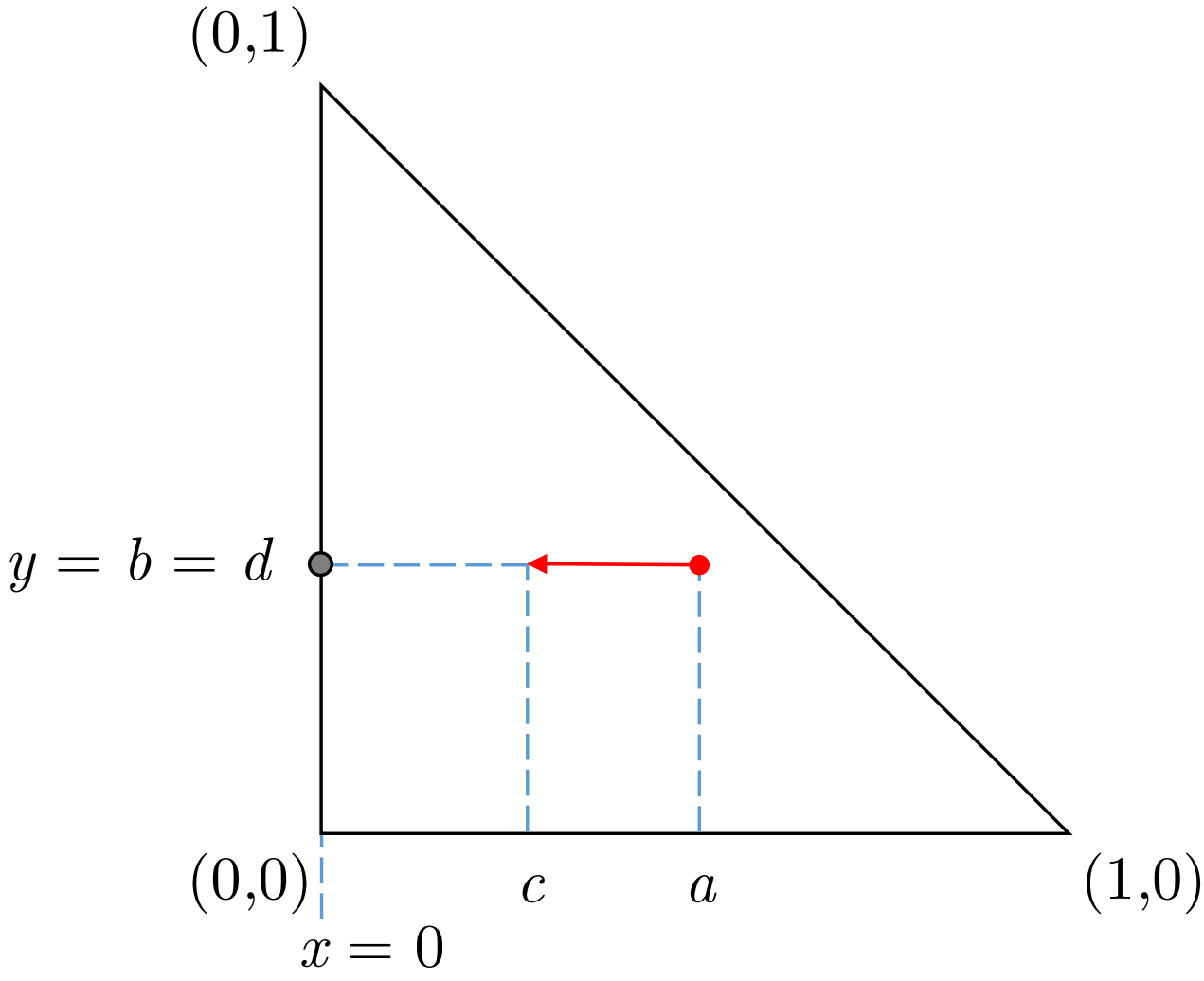


Fig. 14: $a > c, b = d$, (2), (4) and $z = \langle 0, b \rangle$

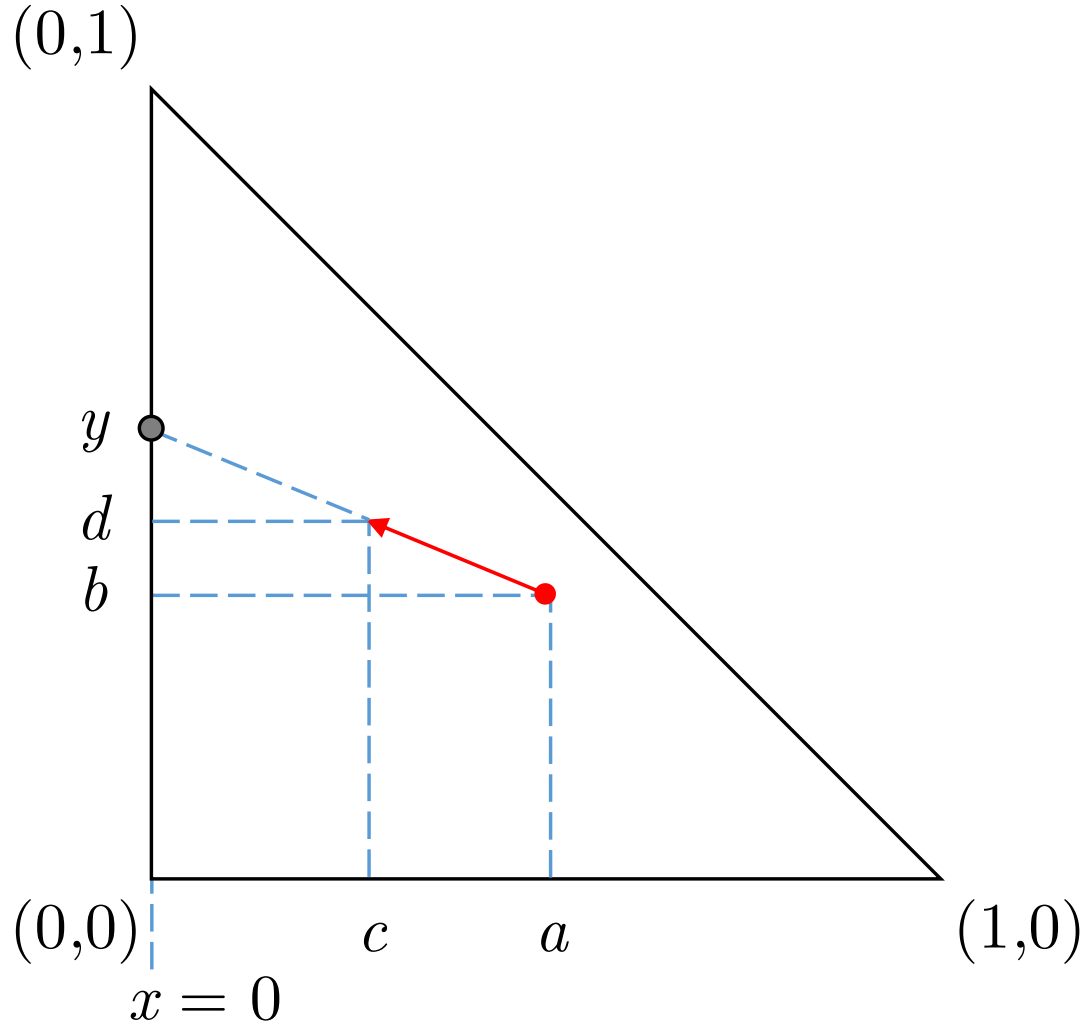


Fig. 15: $a > c, b < d$, (2), (4) and $z = \left\langle 0, \frac{ad-bc}{a-c} \right\rangle$

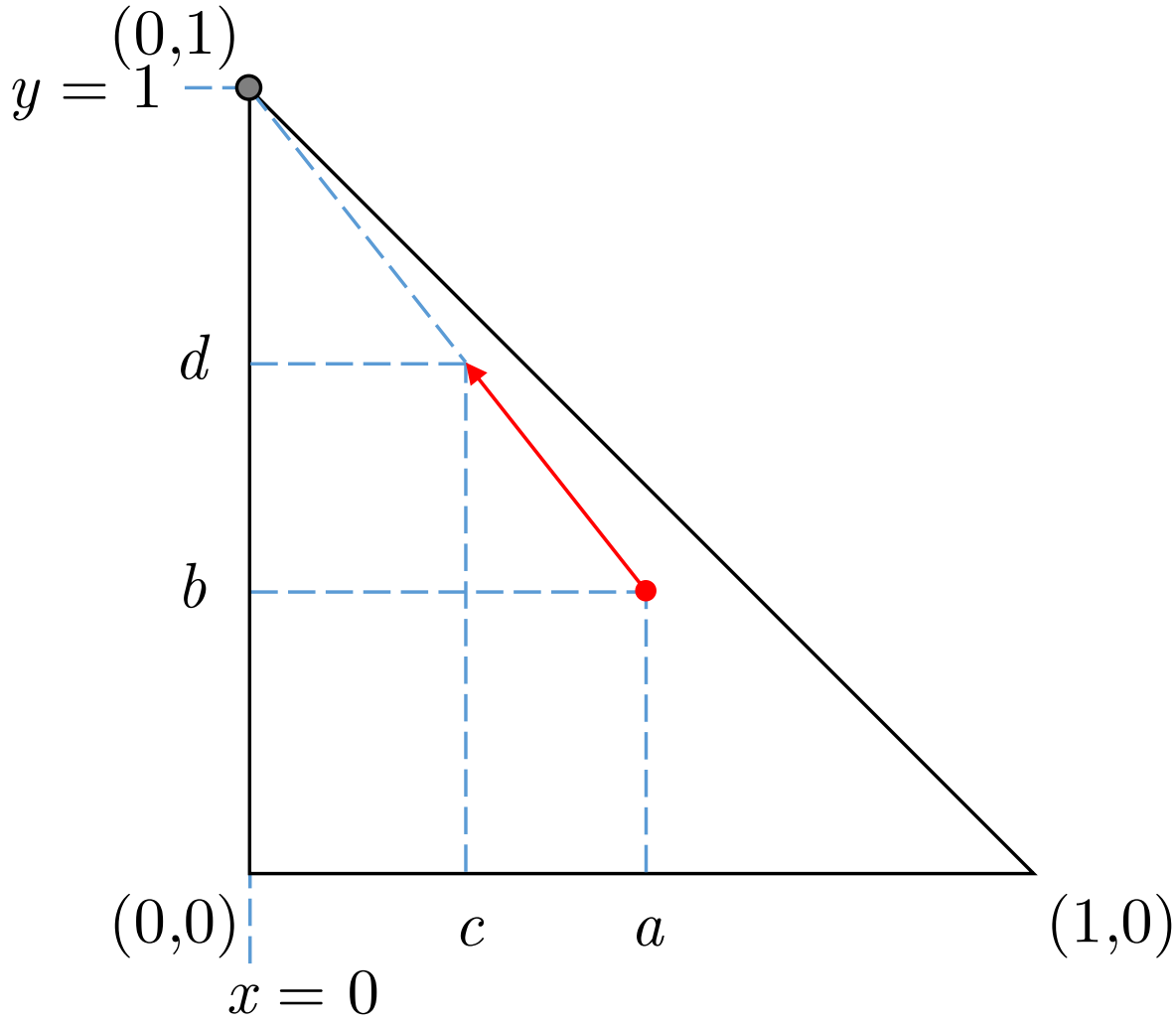


Fig. 16: $a > c, b < d$, $\frac{a}{c} = \frac{1-b}{1-d}$ and $z = \langle 0, 1 \rangle$

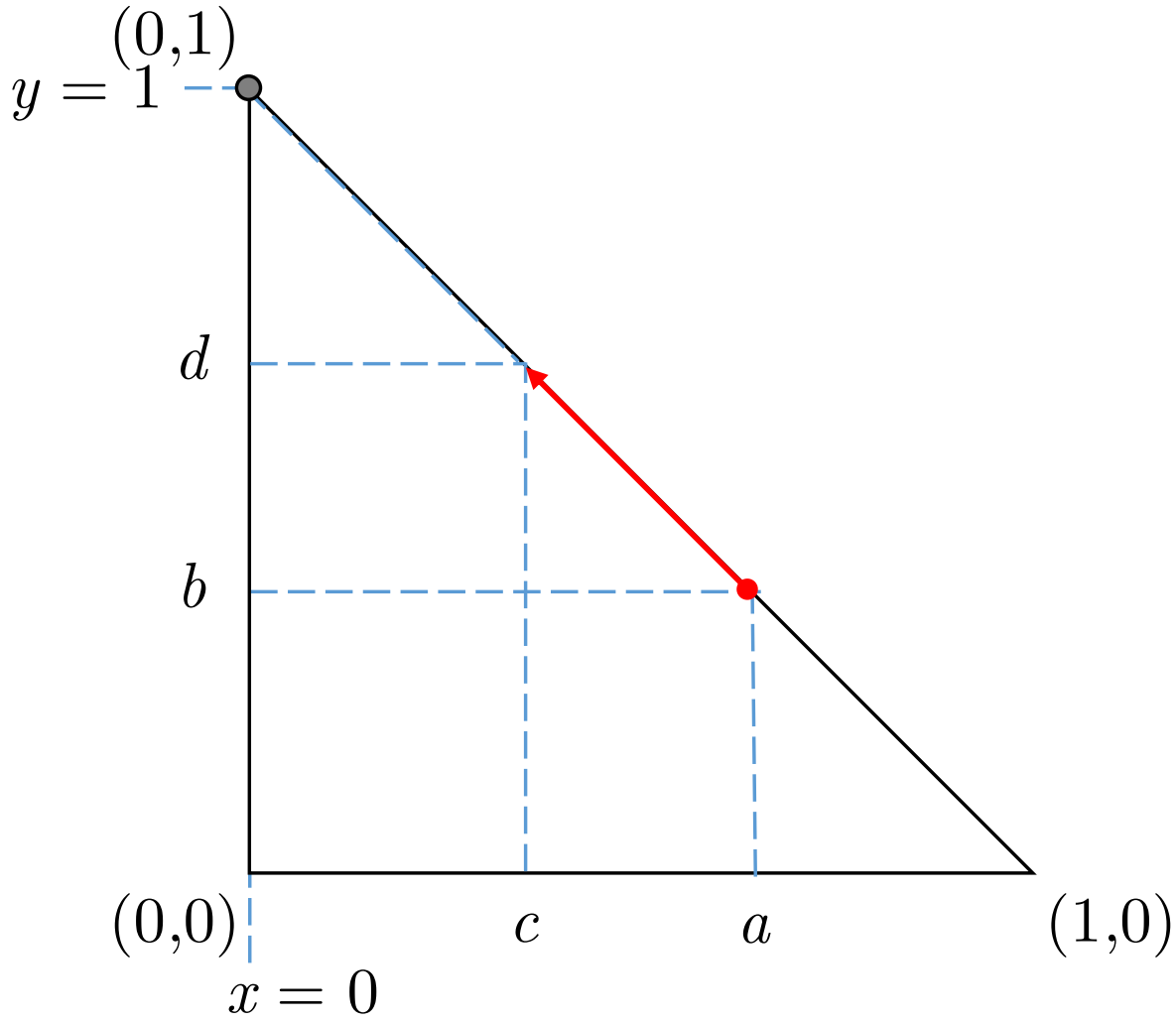


Fig. 17: $a > c, b < d, a + b = c + d = 1$ and $z = \langle 0, 1 \rangle$

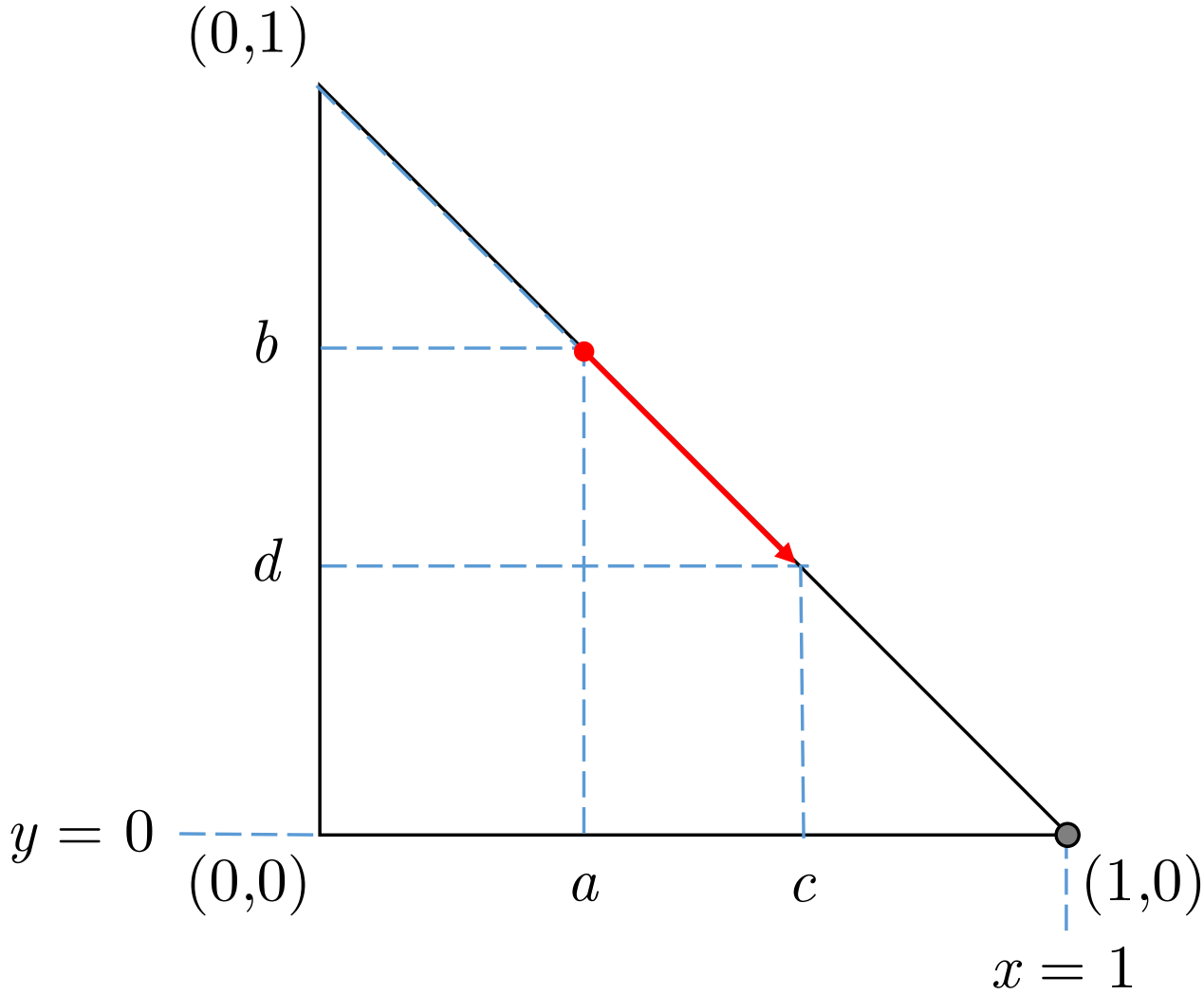


Fig. 18: $a < c, b > d, a + b = c + d = 1$ and $z = \langle 1, 0 \rangle$

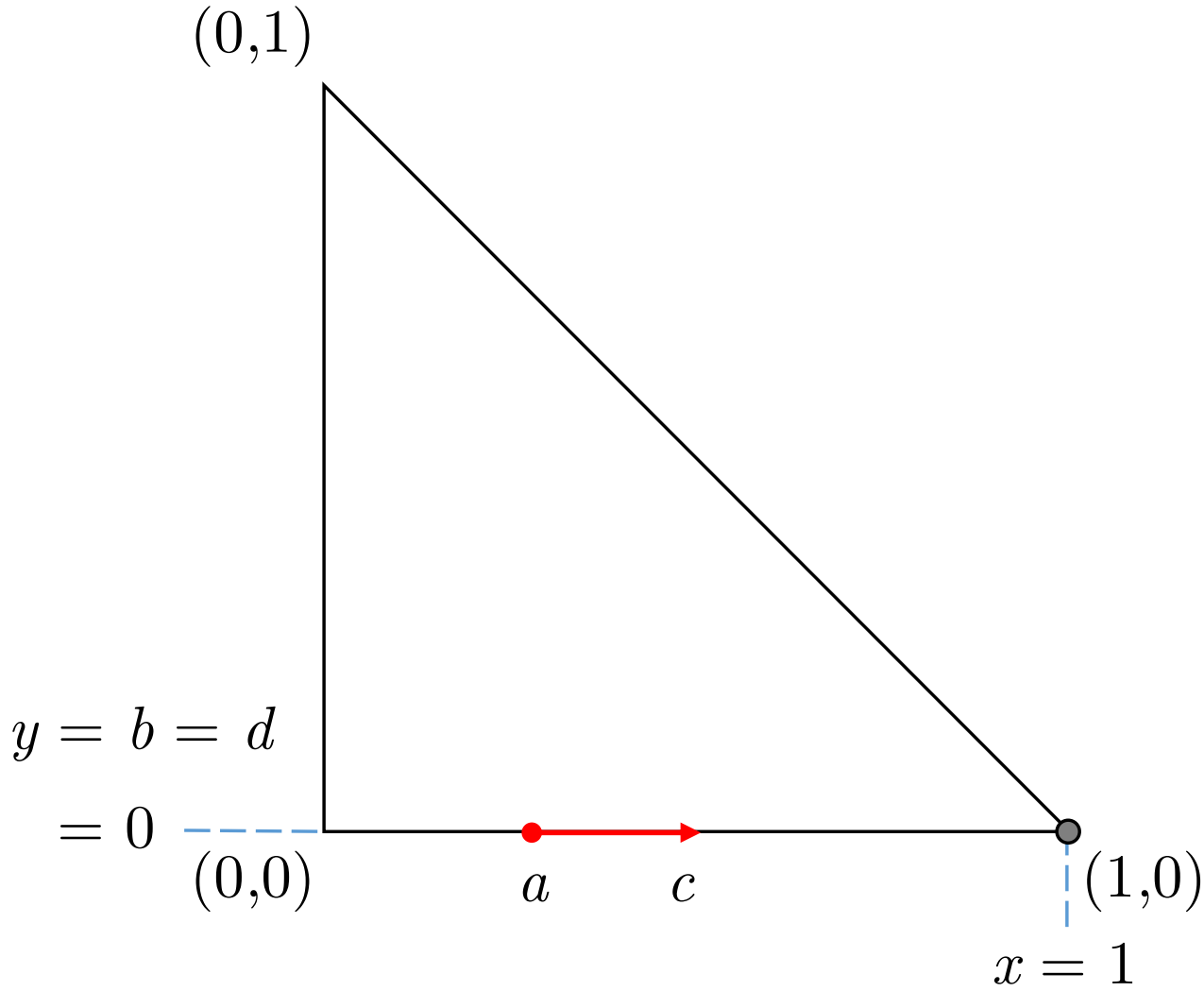


Fig. 19: $a < c, b = d = 0$ and $z = \langle 1, 0 \rangle$

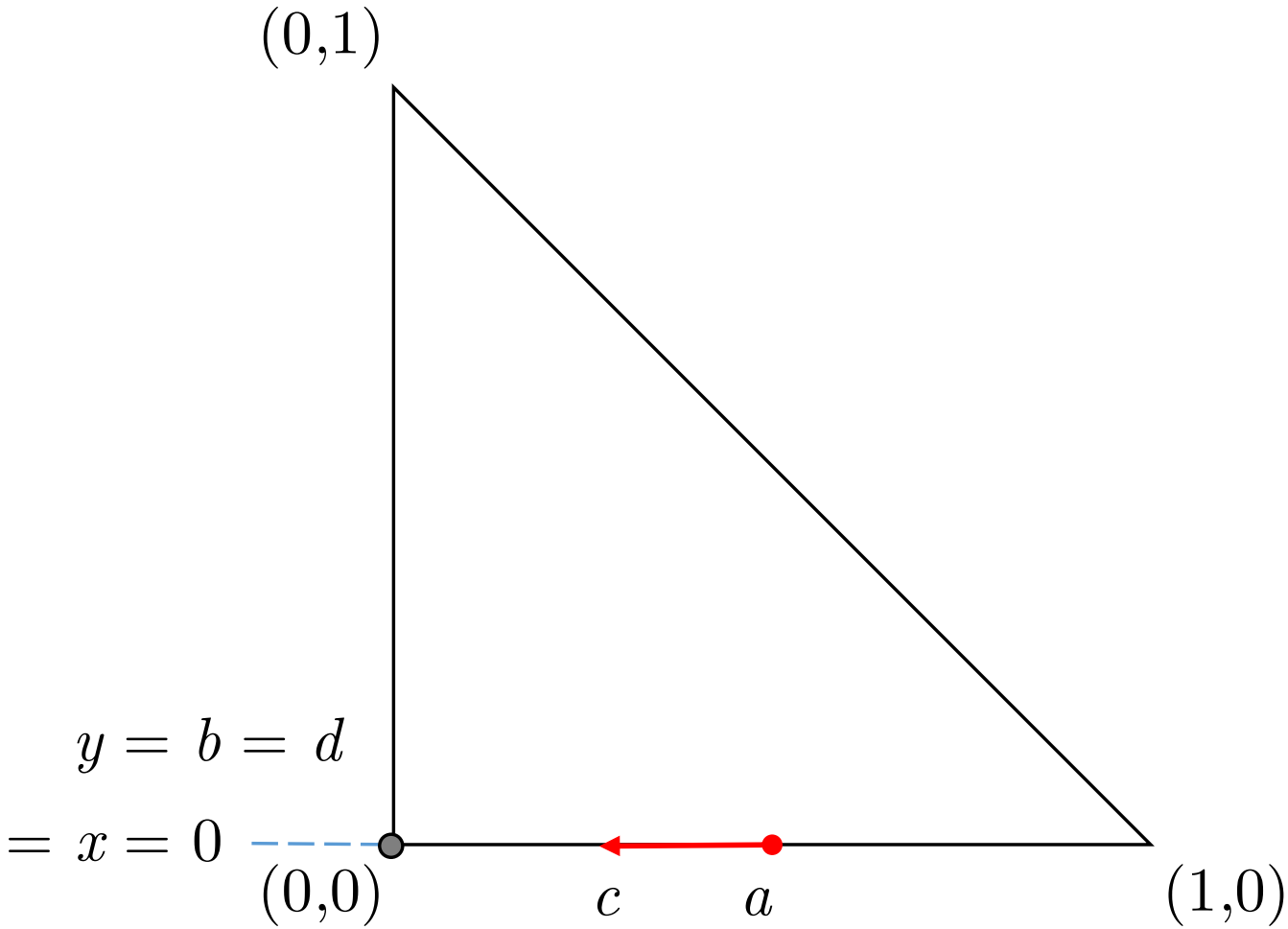


Fig. 20: $a > c, b = d = 0$ and $z = \langle 0, 0 \rangle$

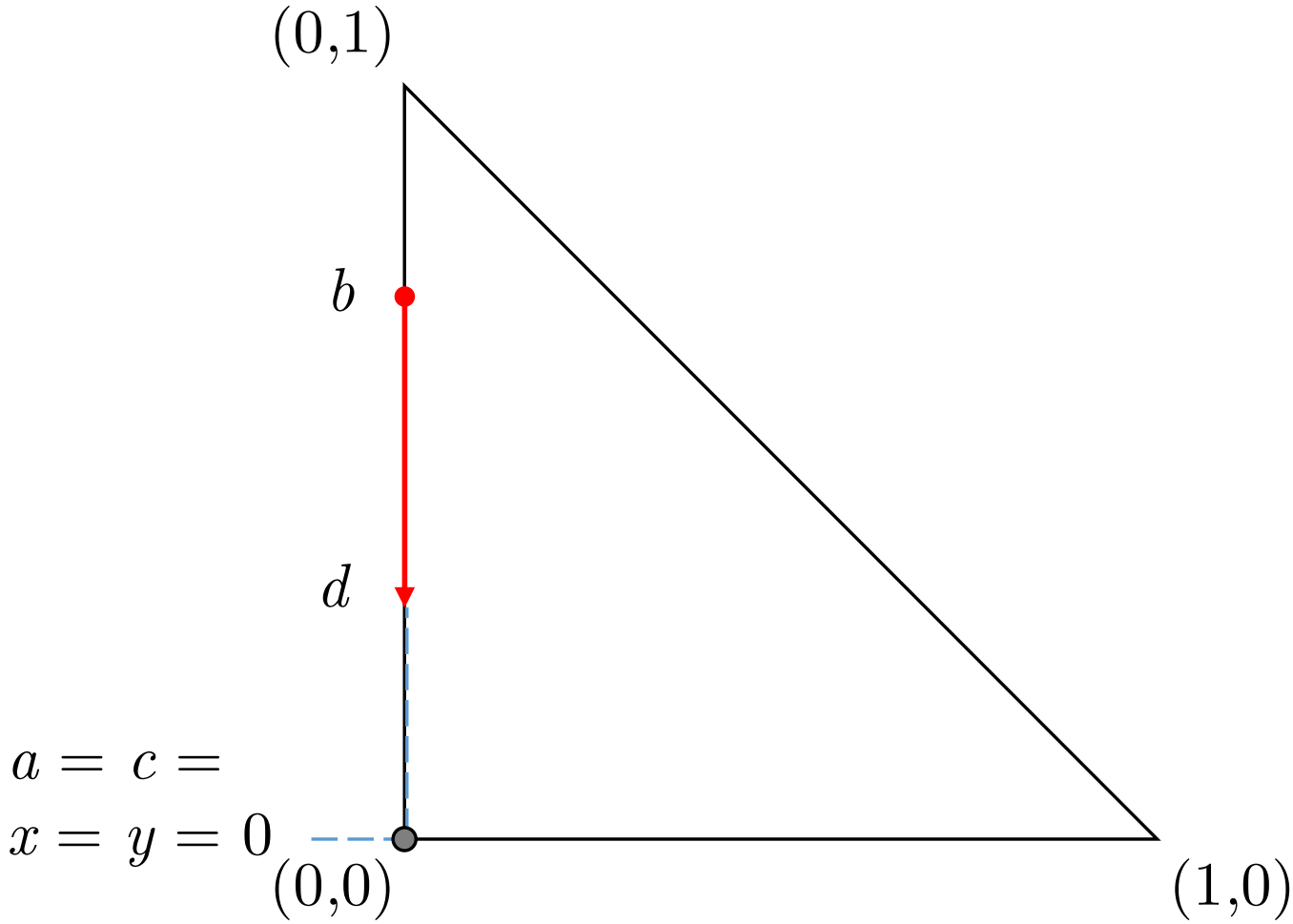


Fig. 21: $a = c = 0, b < d$ and $z = \langle 0, 0 \rangle$

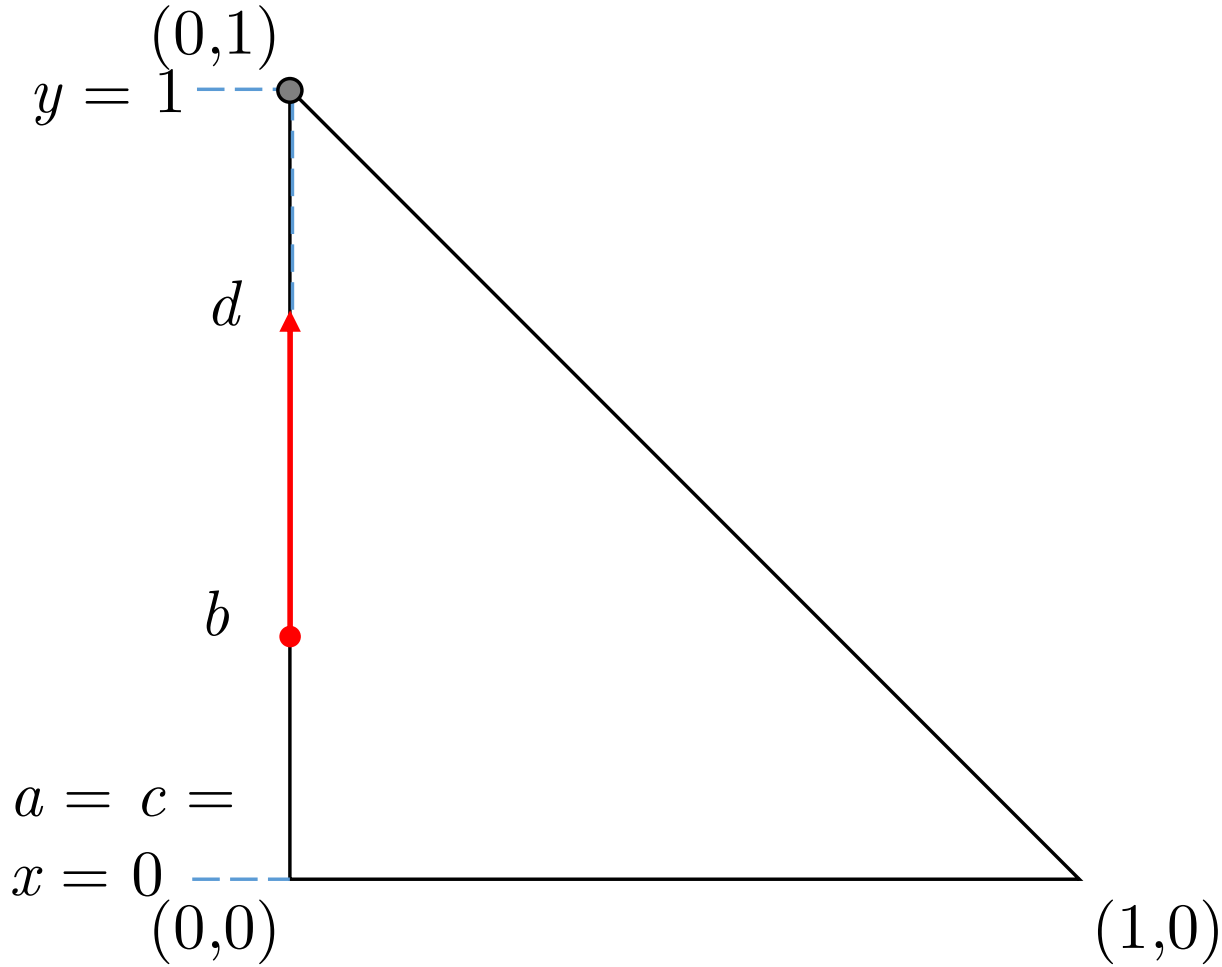


Fig. 22: $a = c = 0, b < d$ and $z = \langle 0, 1 \rangle$

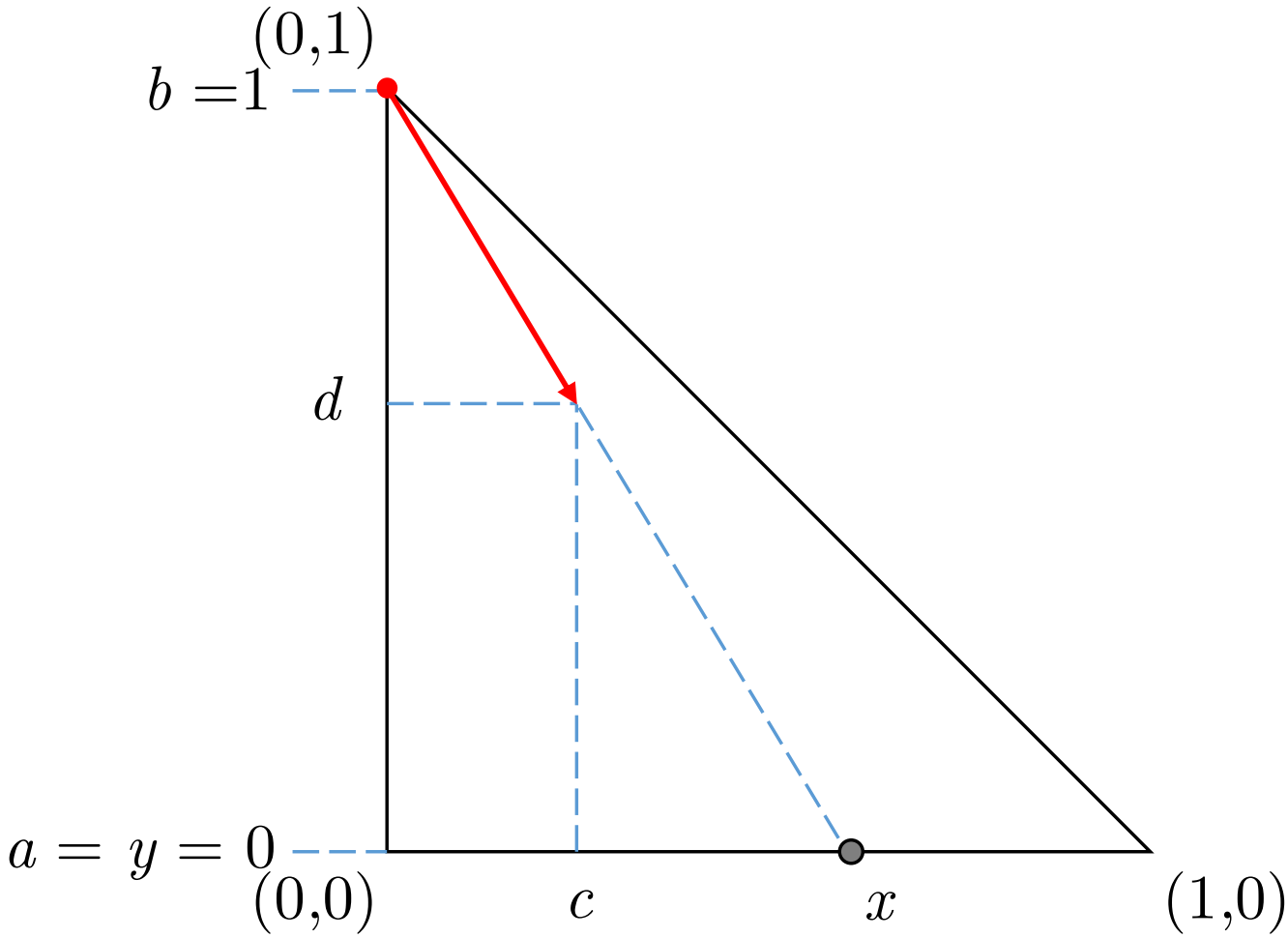


Fig. 23: $a = 0, b = 1$ and $z = \left\langle \frac{c}{1-d}, 0 \right\rangle$

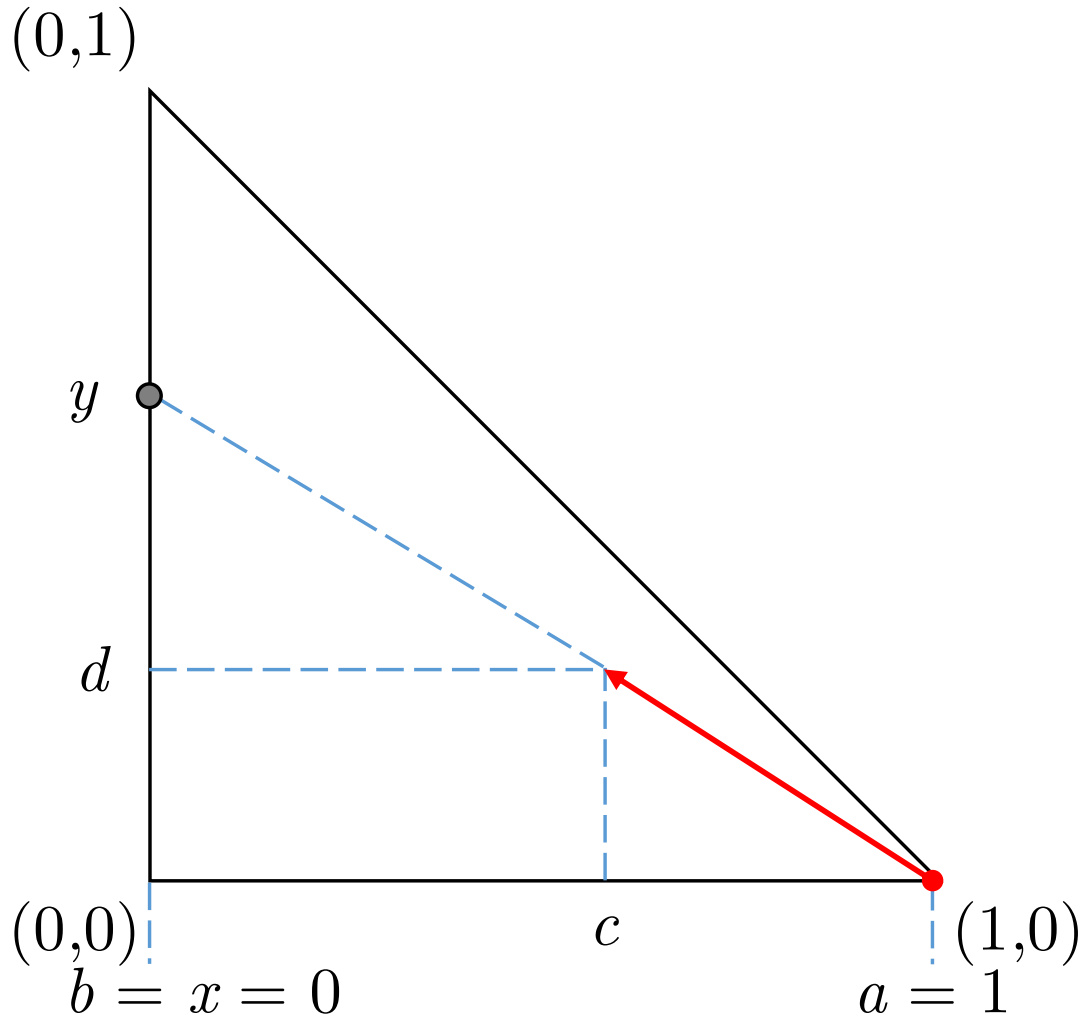


Fig. 24: $a = 1, b = 0$ and $z = \left\langle 0, \frac{d}{1-c} \right\rangle$

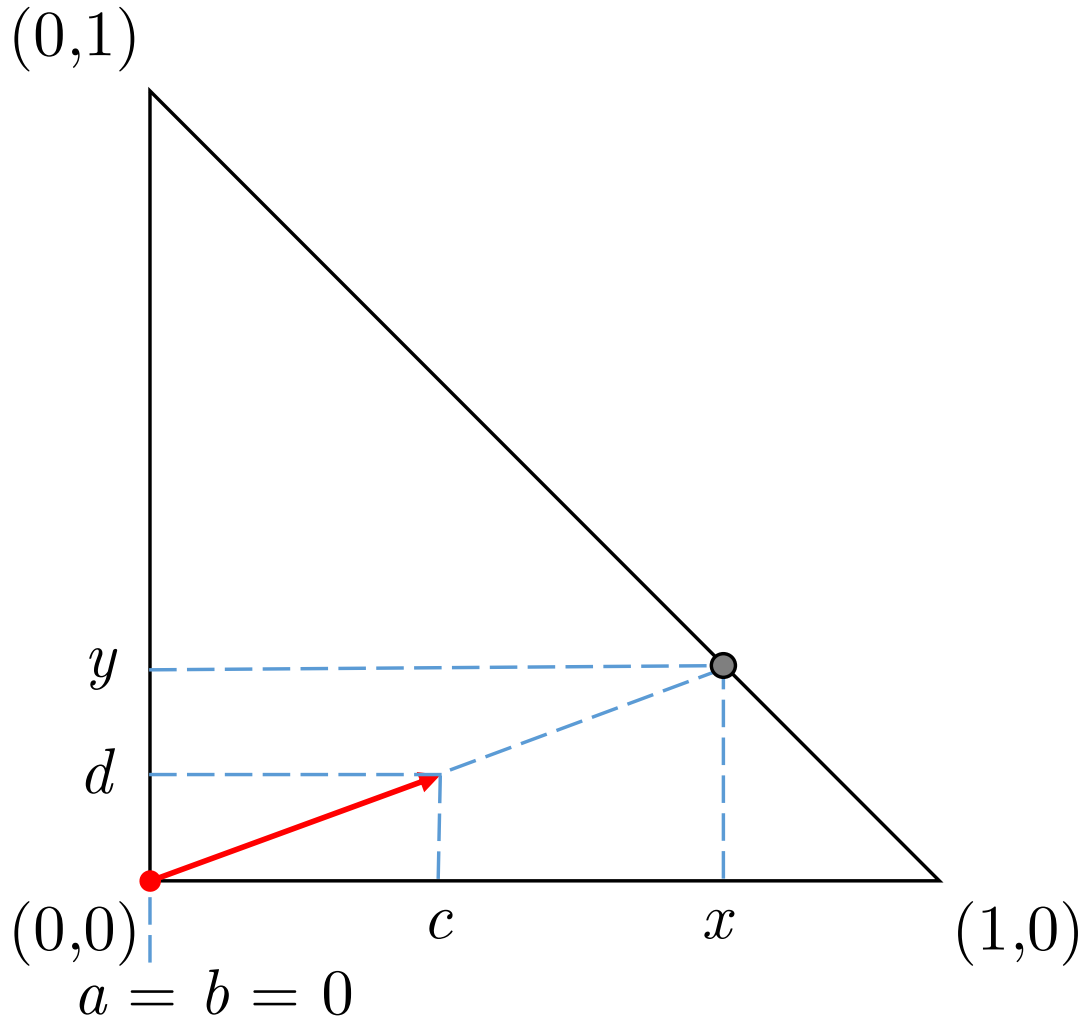


Fig. 25: $a = b = 0$ and $z = \left\langle \frac{c}{c+d}, \frac{d}{c+d} \right\rangle$

From these formulas, as particular cases we see that the following equalities are valid:

- if $a, b, a+b \in (0,1)$ and $c+d=1$, then

$$x * y = \langle c, d \rangle,$$

- if $a, b, a+b \in (0,1)$ and $c \leq 1, d=0$, then

$$x * y = \langle c, 0 \rangle,$$

- if $a, b, a+b \in (0,1)$ and $c=0, d \leq 1$, then

$$x * y = \langle 0, d \rangle,$$

Finally, we must define:

$$\langle a, b \rangle * \langle a, b \rangle = \langle a, b \rangle.$$

Now, we can give an explicit formula for $x * y$:

$$x * y = \begin{cases} \left\langle \frac{c-a+ad-bc}{c+d-a-b}, \frac{d-b-ad+bc}{c+d-a-b} \right\rangle, & \text{if (1) and (2) are valid} \\ \left\langle \frac{bc-ad}{b-d}, 0 \right\rangle, & \text{if (1) and (3) are valid} \\ \left\langle 0, \frac{ad-bc}{a-c} \right\rangle, & \text{if (2) and (4) are valid} \end{cases}$$

Obviously, operation "*" is not commutative, but for it the following assertion is valid.

Theorem 2. For every two IFPs x and y:

$$\neg(x * y) = \neg x * \neg y.$$

Proof: Let the two IFPs x and y satisfy conditions (1) and (2). Then

$$\begin{aligned}
\neg(x * y) &= \neg(\langle a, b\rangle * \langle c, d\rangle) \\
&= \neg \left\langle \tfrac{c-a+ad-bc}{c+d-a-b}, \tfrac{d-b-ad+bc}{c+d-a-b} \right\rangle \\
&= \left\langle \tfrac{d-b-ad+bc}{c+d-a-b}, \tfrac{c-a+ad-bc}{c+d-a-b} \right\rangle \\
&= \langle d, c\rangle * \langle b, a\rangle \\
&= \neg\langle c, d\rangle * \neg\langle a, b\rangle \\
&= \neg x * \neg y.
\end{aligned}$$

The check of the remaining cases is similar. □

We should mention that there is only one other operation over IFPs with this property. It is

$$x@y = \left\langle \frac{a+c}{2}, \frac{b+d}{2} \right\rangle .$$

By direct inspection we can see that it does not satisfy the distributive law with any of the operations "$\wedge$", "$\vee$", "+", ".", "@". Also, we can directly see that the operation "*" is not associative.

Conclusion

In this study, a new operation has been introduced and illustrated.

In the future, other properties of the new operation will be discussed.

Its relationship with some of the modal operators will be shown.

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Thank you for attention!