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A short note on a new ranking for intuitionistic fuzzy alternatives

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Introduction

Intuitionistic fuzzy sets were introduced in 1983 in [1] as an extension of L. Zadeh's fuzzy sets. The modal operators "necessity" ($\Box$) and "possibility" ($\Diamond$), which have no analogues in the fuzzy case, were defined there for the first time. In various decision making problems set in intuitionistic fuzzy setting, experts are tasked with evaluating alternatives with scores in the form of intuitionistic fuzzy pair – which may be viewed as an intuitionistic fuzzy set with a single element x. We will briefly recall some of the notions which will be used further

Definition 1 (cf. [5]).

An intuitionistic fuzzy pair is an ordered couple of non-negative numbers in the unit interval $x = \langle a, b \rangle$, such that:

$$a + b \leq 1. \tag{1}$$

The *hesitancy margin of the IFP* x is the quantity:

$$h(x) = 1 - a - b$$

The geometrical representation of an IFP in the interpretational triangle is given in Fig. 1.

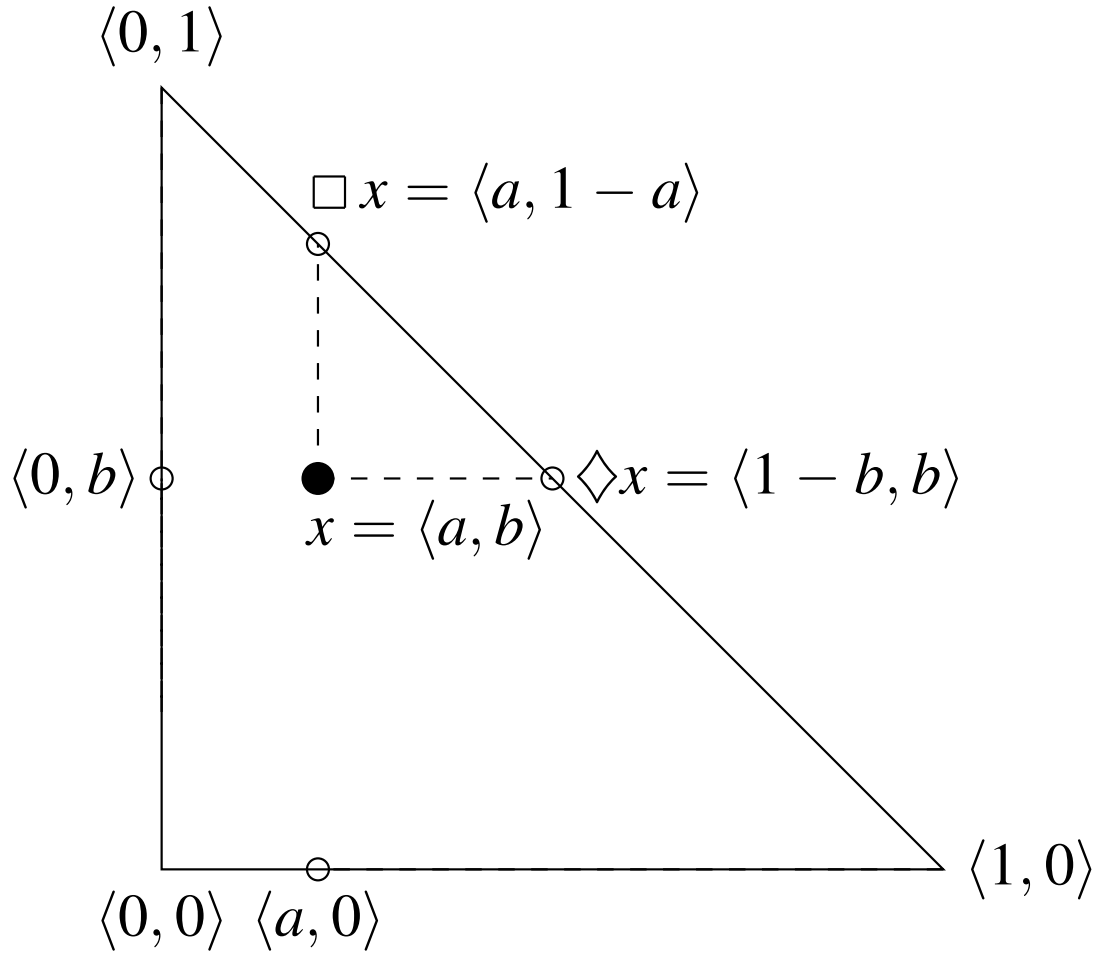


Figure 1: A geometrical interpretation of an IFP $x = \langle a, b \rangle$ and the result of the "necessity" and "posibility" operators

Different orderings have been defined over IFPs. The most recognizable are due to Atanassov [2] and Bustince and Burillo [6] (the latter one has been extensively studied by E. Marinov [7]). Both of these, of course, are partial orderings. Therefore, when two alternatives are incomparable by the chosen ordering the decision maker is faced with a difficult choice. There are several approaches employed most often in practice - the introduction of another ordering – which preserves the order of the other alternatives – and allows the considered alternatives to be compared, the use of a ranking function which juxtaposes a real-valued number to each alternative and permits the linear ordering of the real numbers to be used for selecting the more beneficial one. Another approach is by incorporating some distance based measure reflecting how far the considered alternative x is from the ideal alternative $\langle 1, 0 \rangle$, which also results in real-valued linear order.

The classical partial ordering introduced by Atanassov (see [2]) is given by

Definition 2 (cf. [2,5]).

For two IFPs: $x = \langle x_1, x_2 \rangle$ and $y = \langle y_1, y_2 \rangle$, we say that x is less or equal to y, and we write:

$$x \leq y,$$

iff

$$\begin{cases} x_1 \leq y_1 \\ x_2 \geq y_2. \end{cases} \tag{2}$$

On the other hand, the ordering proposed by Bustince and Burillo is defined by the hesitancy margin of the IFPs.

Definition 3 (cf. [6,7]).

For two IFPs: $x = \langle x_1, x_2 \rangle$ and $y = \langle y_1, y_2 \rangle$, we say that x precedes y (or equivalently, x has greater or equal hesitancy margin than y), and we write:

$$x \preceq y,$$

iff

$$\begin{cases} x_1 \leq y_1 \\ x_2 \leq y_2. \end{cases} \tag{3}$$

Its is quite clear that in some sense the two partial orderings (barring the very particular case when the two IFPs have at least one equal component) are incompatible. If two IFPs are comparable under the first ordering, they are incomparable under the second and vice versa.

In a previous research involving the second and third authors of the present study, a family of orderings depending on the power mean M_p were investigated. We refer the interested reader to [4] for further details. Here we only provide the definition of the power mean based orderings and we mention some of their properties:

Definition 4 ([3]).

Given two IFPs $x = \langle x_1, x_2 \rangle$ and $y = \langle y_1, y_2 \rangle$, we say that x is first component biased power mean-based with a value of p less or equal to y and write $x \preceq_{\mu;M_p} y$ if

$$\begin{cases} 1 - x_1 \geq 1 - y_1 \\ M_p(1 - x_1, x_2) \geq M_p(1 - y_1, y_2). \end{cases} \tag{4}$$

where the power mean of two non-negative numbers x, y is given by

$$M_p(s,t) = \left(\frac{s^p + t^p}{2} \right)^{\frac{1}{p}}.$$

There are special cases (obtained as a limit) of the power mean for particular values of p, which we remind here: $M_{-\infty}(s,t) = \min(s,t)$, $M_0(s,t) = \sqrt{st}$, $M_{\infty}(s,t) = \max(s,t)$.

These family of orderings allow us to compare the element of any set of IFPs which does not contain the $\langle 0, 0\rangle$ alternative, for a suitable choice of p. They also preserve the orderings for all lower values of p. In particular the classical ordering (2) is obtained as (4) for $p = -\infty$. We also remind the following

Definition 5 (cf. [9]).

For two u, v IFPs, the order $\preceq_P$ is defined as follows:

$$u \preceq_P v \Leftrightarrow P(u) \leq P(v) \tag{5}$$

where $P(u) = 1 - M_2(1 - u_1, u_2)$, with M_2 being the power mean of order 2.

Here we will investigate an alternative approach for ranking alternatives in the form of IFPs based on distance measure, and the triangle with vertices $\Box x$, $\Diamond x$, and x.

The proposed ranking function

Let two IFPs $x = \langle a, b \rangle$ and $y = \langle c, d \rangle$ be given. Let the triangle with vertices $\Box x$, $\diamondsuit x$, and x, be denoted by A. Let the triangle with vertices $\Box y$, $\diamondsuit y$, and y, be denoted by B.
We define the following functions

$$\begin{cases} T(x) = \left(\sqrt{2} - \sqrt{(1-a)^2 + b^2}\right)(1 - 2SA) \\ T(y) = \left(\sqrt{2} - \sqrt{(1-c)^2 + d^2}\right)(1 - 2SB), \end{cases} \tag{6}$$

where SA is the area of the triangle A

$$SA = \frac{1}{2}(1 - a - b)^2$$

and SB is the area of the triangle B

$$SB = \frac{1}{2}(1 - c - d)^2.$$

We can simplify the notation by using the denotations $\theta_x = \left(\sqrt{2} - \sqrt{(1-a)^2 + b^2}\right)$, $\theta_y = \left(\sqrt{2} - \sqrt{(1-c)^2 + d^2}\right)$, and rewrite (6) as

$$\begin{cases} T(x) = \theta_x \left(1 - h^2(x)\right) \\ T(y) = \theta_y \left(1 - h^2(y)\right), \end{cases} \tag{6'}$$

Definition 6.

We will say that the alternative x is better than the alternative y under the ranking T and write $x \succ_T y$, if $T(x) > T(y)$. If $T(x) = T(y)$, we say that the alternatives are equivalent with respect to T-ranking. We will write $x \succeq_T y$, to denote the fact that $T(x) \geq T(y)$.

The original idea of the authors involved a different version of the T-ranking, which also included the area of the intersection of the two triangles A and B

$$C(x,y) = S(A \cap B),$$

where if the intersection is a single point or an empty set, the area is taken as 0. In that interpretation the analogue of (6) would look like:

$$\begin{cases} T^*_\lambda(x,y) = \left(\sqrt{2} - \sqrt{(1-a)^2 + b^2}\right)(1 - 2SA + \lambda C(x,y)) \\ T^*_\lambda(y,x) = \left(\sqrt{2} - \sqrt{(1-c)^2 + d^2}\right)(1 - 2SB + \lambda C(x,y)), \end{cases} \tag{7}$$

where $\lambda \in [0,1]$. This variation, however, ruins the transitivity in general.

Remark 1.

The proposed ranking T is not always compatible with the classical order in (2). *If $x \geq y$, and $h(x) \leq h(y)$ they are compatible. Indeed,* (2) *implies $\theta_x \geq \theta_y$, and since $1 - 2SA = 1 - h^2(x)$,*
$1 - 2SB = 1 - h^2(y)$, *hence in this case $x \succeq_T y$.*

A concrete counterexample is $x = \langle 0.50, 0.05 \rangle, y = \langle 0.45, 0.25 \rangle$, we have $x \geq y$, however, $T(x) = 0.727097$ and $T(y) = 0.737156$

Proposition 1.

The ranking T has the following relation with the ranking P proposed in [9] (see Definition 5 above). They coincide for all IFPs x, y for which $h(x) = h(y)$.

Proof.

We can rewrite θ_x as

$$\theta_x = \sqrt{2}\left(1 - \sqrt{\frac{(1-a)^2 + b^2}{2}}\right) = \sqrt{2}\,(1 - M_2(1-a, b)) = \sqrt{2}P(x),$$

hence for pairs such that $0 \leq h(x) = h(y) = h < 1$ we have:

$$T(x) = \sqrt{2}(1-h)P(x); T(y) = \sqrt{2}(1-h)P(y).$$

If $h = 1$, then $x = y = \langle 0, 0 \rangle$. □

One possible venue, in view of Proposition 1, to extend these rankings is to consider a family of rankings depending on the power mean of order $p > 0$, T_p, defined as:

$$\begin{cases} T_p(x) = 2^{\frac{1}{p}} \left(1 - M_p(1-a, b)\right) \left(1 - h^p(x)\right) \\ T_p(y) = 2^{\frac{1}{p}} \left(1 - M_p(1-c, d)\right) \left(1 - h^p(y)\right). \end{cases} \tag{8}$$

and by analogy, $x \succ_{T_p} y$ would be true when $T_p(x) > T_p(y)$. The transitivity of the T_p ranking is straightforwardly verified. For the special case $p = 1$, we obtain $T_1(x) = (1 + a - b)(a + b)$, which somewhat resembles the ranking proposed by Guo [8] induced by:

$$Z(x) = \frac{1}{4}(1 + a - b)(1 + a + b).$$

For the special case $T_2(x)$, we recover our T-ranking. For the special case $p \to +\infty$, we obtain

$$\lim_{p \to +\infty} 2^{\frac{1}{p}} = 1$$

$$M_\infty(1-a, b) = \max(1-a, b) = 1-a$$

For $h(x) < 1$

$$\lim_{p \to +\infty} 1 - h^{\frac{1}{p}}(x) = 1$$

Thus,

$$\lim_{p \to +\infty} T_p(x) = a$$

For $h(x) = 1$, we have $x = \langle 0, 0 \rangle$, hence the equality is also fulfilled in this case.

We proposed a new ranking method inspired by the necessity and possibility enveloping the IFP. We showed how it is related to some other ranking methods. Also, we extended this ranking to one depending on the power mean M_p with $p > 0$. In the future, we will investigate whether values for $p < 0$ are suitable as well as look into other possibilities for extensions.

In the meantime, together with Assoc. Prof. Vassia Atanassova, we have developed the following prototype exploratory tool for ranking alternatives implementing the proposed ranking (which we plan to expand with additional methods proposed in the scientific literature) IFS Ranking Geometry Explorer, which we believe will be of great help to any decision maker.

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