

Stability of a Simple Bioreactor Model with a General Growth Function, Bacterial Feed and Death Rate

Andrew J. Stacey*, John J. Shepherd

School of Science
RMIT University
Melbourne Australia
E-mails: john.shepherd@rmit.edu.au
andrew.stacey@rmit.edu.au

*Corresponding author

Received: January 16, 2021

Accepted: February 20, 2022

Published: December 31, 2022

Abstract: Biological reactors are employed in industrial applications to break down organic waste. Here we consider the stability of an open loop CSTR, with a general growth rate function, having both a bacterial and substrate feed and a death rate. Reactors with a bacterial feed are important as every reactor after the first in a cascade will have both bacterial and substrate feeds. We show that stability of a critical point is related to the sign of the slope of the growth rate function, when no bacterial feed is present, and to the sign of the slope of a function, being the product of the growth rate function and a function involving the substrate and bacterial feed concentrations, when a bacterial feed is present.

Keywords: Stability, Reactor, Bacterial feed, General growth function.

Introduction

Reactors are used extensively in industry for the treatment of waste and the literature analysing the stability of different reactor models is vast. The references contained in the texts [1, 4, 9] give a small sample.

Although an influent bacterial feed is seldom used in models of a single reactor it is always present in models of a cascade of reactors where the output of the previous reactor constitutes the input to the next reactor in the cascade. Results for commonly used growth functions are well known. For example the stability analysis for models incorporating the Haldane growth function for a reactor with a bacterial inflow and death rate is included in Bertuccio et al. [2] and references therein.

While there are results for these frequently used growth rate functions, such as Haldane and Monod, some authors [3, 5, 6, 8] have argued that these are too restrictive and there is a need to consider general classes of growth rate functions in order to identify unifying features.

Lara-Cisneros et al. [6] for example consider a general class of growth rate functions, $F(S)$, of substrate concentration S , but do not include a bacterial feed or death rate. Simeonov and Diop [8] consider a class of growth rate functions, $F(S, X)$, that also includes the bacterial concentration X , (an example of this is the Contois growth function) but again omit a bacterial feed and death rate. Calderon et al. [3] consider an even bigger class of growth rate functions, $F(S, X)$, specific to a system of three equations governing autocatalytic reactions, their aim being to maximise yield as a control goal. Dramé et al. [5] consider a cascade of reactors with a general growth rate function $F(S)$. They include an influent bacterial flow in their first reactor as well as in subsequent reactors in the cascade. They however do not include a death rate.

Their aim is to highlight the existence of multiple critical points in the different reactors in the cascade.

The main contribution of this paper is to consider the stability of a single reactor with both a bacterial feed and a death rate, for a broad class of growth rate functions $F(S)$. We show that stability of a critical point is related to the sign of the slope of the growth rate function, when no bacterial feed is present, and to the sign of the slope of the growth rate function multiplied by a function involving both the substrate and bacterial feed concentrations, when a bacterial feed is present.

Model

The model considered here is of a single continuously stirred tank reactor without recycle, of volume V , with a continuous volume flow rate Q , and constant influent substrate and bacterial feed concentrations, S_{in} and X_{in} respectively, as shown in Fig. 1.

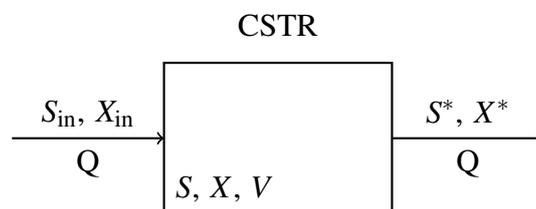


Fig. 1 A single reactor with both substrate and bacterial inputs, S_{in} and X_{in} , a volume flow rate Q and tank volume V .
The output – substrate and bacterial concentrations, S^* and X^* , when operated at a steady state (S^*, X^*) .

$S(T)$ and $X(T)$ are the concentrations of substrate and bacteria in the reactor at time T . The dilution rate is $D = Q/V$, while the hydraulic residence time, the mean time a substrate or bacterium particle is in the reactor, is the inverse of the dilution rate, $1/D$. The reaction is governed by a general bacterial growth rate function $F(S)$, and a constant death rate, $K_d \geq 0$.

The dynamics of the reactor is governed by:

$$\frac{dS}{dT} = -D(S - S_{in}) - \frac{F(S)X}{Y}, \quad (1)$$

$$\frac{dX}{dT} = -D(X - X_{in}) - K_d X + F(S)X, \quad (2)$$

where the maximum specific growth rate, considered constant, has been incorporated into the growth rate function $F(S)$.

In what follows, we assume that $F(S)$ meets the following conditions.

Definition. The growth rate function $F(S)$ is continuous and differentiable, $F(0) = 0$, $F(S) > 0$ and is bounded on $S > 0$.

This definition includes both the Monod and Haldane growth rate functions, as well as functions with multiple maxima.

Eqs. (1) and (2) can be simplified if the bacterial concentrations X and X_{in} are scaled by the constant yield coefficient Y , to give $\bar{X} = X/Y$ and $\bar{X}_{in} = X_{in}/Y$. In what follows we will drop the overbar on the X .

The system of Eqs. (1) and (2) then become

$$\frac{dS}{dT} = -D(S - S_{in}) - F(S)X, \quad (3)$$

$$\frac{dX}{dT} = -D(X - X_{in}) - K_d X + F(S)X. \quad (4)$$

Then, $S(T)$ and $X(T)$ are the substrate and scaled bacterial concentrations respectively, at time T . The substrate and scaled bacterial concentrations entering the reactor are assumed to be constant which makes the system of Eqs. (3) and (4) autonomous. Here, we assume $S_{in} > 0$, $X_{in} \geq 0$, and $K_d \geq 0$. In what follows we refer to the scaled bacterial concentrations simply as the bacterial concentrations.

Reactor critical points: $X_{in} > 0$

The critical points (CPs) of the reactor, (S^*, X^*) , are solutions of Eqs. (3) and (4) where the right hand sides are set to zero, that is, they are solutions of the system of two equations

$$0 = -D(S - S_{in}) - F(S)X, \quad (5)$$

$$0 = -D(X - X_{in}) - K_d X + F(S)X. \quad (6)$$

We first find X in terms of S by adding Eqs. (5) and (6) and solving for X to get

$$X = \frac{D(S_{in} + X_{in} - S)}{B}, \quad \text{where } B = D + K_d. \quad (7)$$

Substituting Eq. (7) into Eq. (5) gives the equation that S^* must satisfy,

$$B = R(S), \quad (8)$$

where

$$R(S) = \frac{S_{in} + X_{in} - S}{S_{in} - S} F(S).$$

There will in general be more than one solution to Eq. (8), depending on the functional form of $F(S)$. The corresponding X^* can now be found using Eq. (7).

We note that the system cannot washout since we assume $X_{in} > 0$. As a consequence $S^* < S_{in}$ at a physically realistic CP. To be physically realistic we require $S^* > 0$, $X^* \geq 0$.

Looking at the function $R(S)$ we note that it is continuous on the interval $(0, S_{in})$, with a vertical asymptote at $S = S_{in}$, so there will be at least one real solution of Eq. (8) in the interval $(0, S_{in})$ for any $B > 0$.

Graphically the S^* values are the points of intersection of the two functions $B = D + K_d$ and $R(S)$ (see Fig. 2). We can interpret the S^* values as points of balance, where the rate of live bacteria being removed from the reactor, B , (which includes by death, given by K_d , and being

flushed out, given by D) is balanced by the rate of increase of new bacteria in the reactor, given by $R(S^*)$. Note that this last term incorporates both the rate of growth of bacteria due to the conversion of substrate, given by $F(S)$, as well as the rate of inflow of new bacteria given by X_{in} .

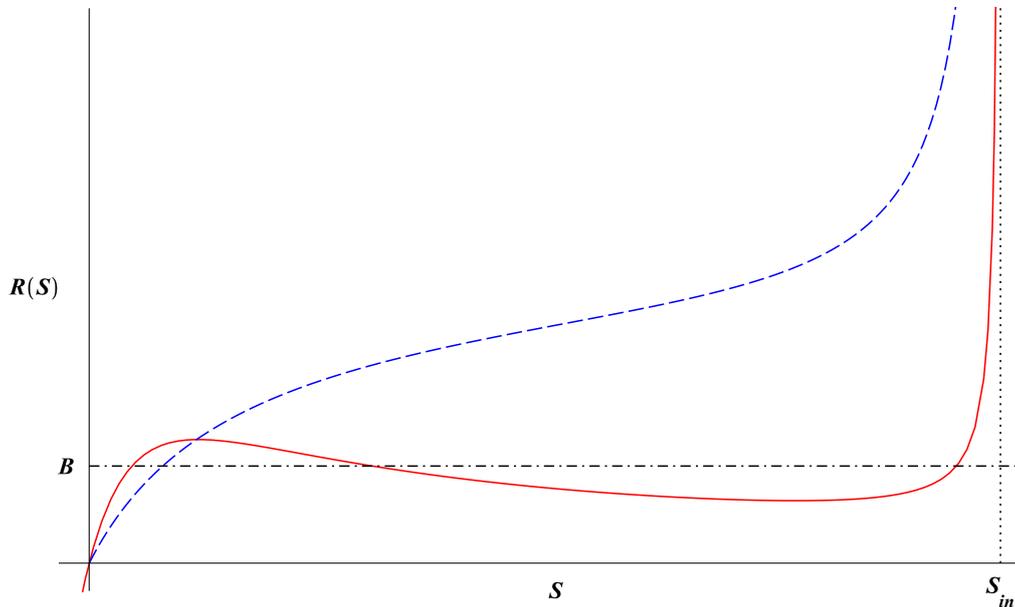


Fig. 2 The red (solid) plot shows an example of $R(S)$ for a Haldane growth function, while the blue (dashed) curve shows an example of a Monod growth function. The horizontal black (dot dash) line is an example B value.

Reactor stability: $X_{in} > 0$

The stability of the physically realistic CPs is determined by looking at the eigenvalues of the linearised system Eqs. (3) and (4) evaluated at each of the critical points (S^*, X^*) . The Jacobian matrix of the right hand sides of Eqs. (3) and (4), at a critical point (S^*, X^*) is

$$J(S^*, X^*) = \begin{bmatrix} -D - F'(S^*)X^* & -F(S^*) \\ F'(S^*)X^* & -B + F(S^*) \end{bmatrix},$$

where the prime indicates a derivative taken with respect to S . The determinant and trace of $J(S^*, X^*)$ are given by,

$$\begin{aligned} \det J(S^*, X^*) &= D(B - F(S^*)) + BF'(S^*)X^*, \\ \text{tr} J(S^*, X^*) &= -(B - F(S^*)) - D - F'(S^*)X^*. \end{aligned}$$

The eigenvalues of $J(S^*, X^*)$ are then given by

$$\lambda_{1,2} = \frac{\text{tr} J(S^*, X^*) \pm \sqrt{\text{tr} J(S^*, X^*)^2 - 4 \det J(S^*, X^*)}}{2}.$$

A CP will be asymptotically stable (AS) when both eigenvalues have negative real parts, that is when $\text{tr} J(S^*, X^*) < 0$ and $\det J(S^*, X^*) > 0$. If in addition $\det J(S^*, X^*) > \text{tr} J(S^*, X^*)^2$ the point will be a *spiral point*, otherwise it will be a *stable node*. The critical point, (S^*, X^*) , will be a *saddle point* if the eigenvalues are real and of opposite sign.

Theorem. For all critical points (S^*, X^*) ,

$$\det J(S^*, X^*) = D(S_{in} - S^*)R'(S^*),$$

where $R(S) = \frac{S_{in} + X_{in} - S}{S_{in} - S}F(S)$ and $S^* \in (0, S_{in})$.

Three cases exist:

1. $\det J(S^*, X^*) > 0$, then $\text{tr} J(S^*, X^*) < 0$, and (S^*, X^*) is an AS CP;
2. $\det J(S^*, X^*) < 0$, then (S^*, X^*) is a saddle point;
3. $\det J(S^*, X^*) = 0$, then (S^*, X^*) is a saddle node bifurcation point.

Proof. To see that $\det J(S^*, X^*) = D(S_{in} - S^*)R'(S^*)$, we first need to show that

$$F'(S^*)(S_{in} + X_{in} - S^*) = (S_{in} - S^*)R'(S^*) - \frac{X_{in}}{S_{in} - S^*}F(S^*).$$

Now,

$$R'(S) = \frac{X_{in}}{(S_{in} - S)^2}F(S) + \frac{S_{in} + X_{in} - S}{S_{in} - S}F'(S), \quad (9)$$

so multiplying through by $(S_{in} - S)$, rearranging and evaluating at $S = S^*$ gives the result.

Now consider,

$$\begin{aligned} \det J(S^*, X^*) &= D(B - F(S^*)) + BF'(S^*)X^* \\ &= D(B - F(S^*)) + DF'(S^*)(S_{in} + X_{in} - S^*) \\ &= D[(B - F(S^*)) + F'(S^*)(S_{in} + X_{in} - S^*)]. \end{aligned}$$

Since $B = \frac{S_{in} + X_{in} - S^*}{S_{in} - S^*}F(S^*)$ at a critical point then

$$\begin{aligned} B - F(S^*) &= \left[\frac{S_{in} + X_{in} - S^*}{S_{in} - S^*} - 1 \right] F(S^*) \\ &= \frac{X_{in}}{S_{in} - S^*}F(S^*), \end{aligned}$$

so

$$\begin{aligned} \det J(S^*, X^*) &= D \left[\frac{X_{in}}{S_{in} - S^*}F(S^*) + (S_{in} - S^*)R'(S^*) - \frac{X_{in}}{S_{in} - S^*}F(S^*) \right] \\ &= D(S_{in} - S^*)R'(S^*). \end{aligned}$$

The three cases follow:

1. Assume $\det J(S^*, X^*) > 0$ then

$$\begin{aligned} \det J(S^*, X^*) &= D(B - F(S^*)) + BF'(S^*)X^* > 0, \quad \text{or} \\ &-F'(S^*)X^* < \frac{D}{B}(B - F(S^*)). \end{aligned}$$

It follows that

$$\begin{aligned}\operatorname{tr}J(S^*, X^*) &= -(B - F(S^*)) - D - F'(S^*)X^*, \\ &< -(B - F(S^*)) - D + \frac{D}{B}(B - F(S^*)) \\ &= -(B - F(S^*)) \left(1 - \frac{D}{B}\right) - D < 0,\end{aligned}$$

so $\operatorname{tr}J(S^*, X^*) < 0$ and (S^*, X^*) is an AS CP.

2. If $\det J(S^*, X^*) < 0$, irrespective of the sign of $\operatorname{tr}J(S^*, X^*)$, both eigenvalues will be real and have opposite signs, so the CP, (S^*, X^*) , is a saddle point.
3. When $\det J(S^*, X^*) = 0$ exactly one eigenvalue is zero and the other is real, so (S^*, X^*) is a saddle node bifurcation point.

□

We note that, starting from the smallest S^* value, the stability of the corresponding critical points alternates, the odd points being asymptotically stable critical points and the even points being unstable saddle points.

Reactor critical points: $X_{in} = 0$

Although the results in this section are not new it is worth reconsidering them in the framework given above. We note that, compared to some authors [5], the growth rate functions considered here have fewer restrictions.

When $X_{in} = 0$, Eqs. (5) and (6) reduce to

$$0 = -D(S - S_{in}) - F(S)X, \quad (10)$$

$$0 = -DX - K_d X + F(S)X. \quad (11)$$

Adding Eq. (10) and Eq. (11) and solving for X gives

$$X = \frac{D(S_{in} - S)}{B} \quad \text{where} \quad B = D + K_d.$$

Substituting this back into Eq. (10) gives the equation that S^* must satisfy at a CP,

$$D(S_{in} - S) \left(1 - \frac{F(S)}{B}\right) = 0.$$

We see that $S^* = S_{in}$ is always solution for any B . The other solutions satisfy $B = F(S^*)$. In Fig. 3 we see that, for a given B value, the S^* values for the critical points are represented as the points of intersection of the constant B with $F(S)$ and B with $S = S_{in}$. For the value of B depicted, there are four non-washout S^* and one washout S^* .

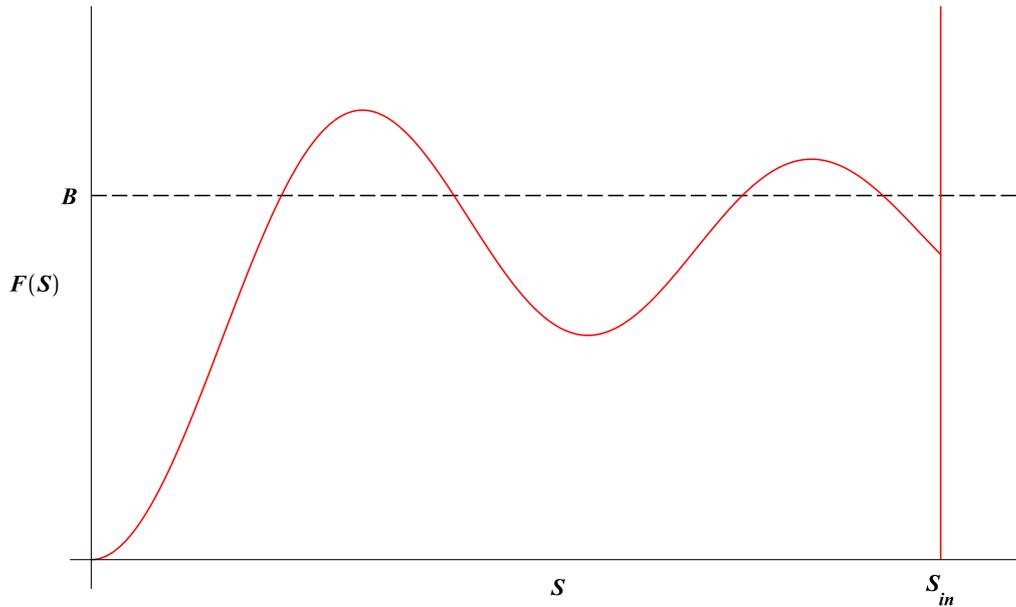


Fig. 3 The red (solid) plot shows an example $F(S)$. The red (solid) vertical line at $S^* = S_{in}$ gives one of the solutions for any B , corresponding to washout CPs. The other S^* solutions are given by the intersection of the black (dashed) line and $F(S)$, with the number depending on the particular B value.

Reactor stability: $X_{in} = 0$

Corollary. For all critical points (S^*, X^*) where $B = F(S^*)$,

$$\det J(S^*, X^*) = D(S_{in} - S^*)F'(S^*),$$

$$\text{tr} J(S^*, X^*) = -D - \frac{\det J(S^*, X^*)}{B},$$

where $S^* \in (0, S_{in})$.

Three cases exist:

1. $\det J(S^*, X^*) > 0$, when $F'(S) > 0$, then $\text{tr} J(S^*, X^*) < 0$, and (S^*, X^*) is an AS CP;
2. $\det J(S^*, X^*) < 0$, then (S^*, X^*) is a saddle point;
3. $\det J(S^*, X^*) = 0$, then (S^*, X^*) is a saddle node bifurcation point.

For all washout critical points, $(S_{in}, 0)$,

$$\det J(S_{in}, 0) = D(B - F(S_{in})),$$

$$\text{tr} J(S_{in}, 0) = -D - \frac{\det J(S_{in}, 0)}{D}.$$

Three cases exist:

1. $\det J(S_{in}, 0) > 0$, when $B > F(S_{in})$, then $\text{tr} J(S_{in}, 0) < 0$, and $(S_{in}, 0)$ is an AS CP;

2. $\det J(S_{\text{in}}, 0) < 0$, when $B < F(S_{\text{in}})$, then $(S_{\text{in}}, 0)$ is a saddle point;
3. $\det J(S_{\text{in}}, 0) = 0$, then $(S_{\text{in}}, 0)$ is a saddle node bifurcation point.

We see, in the $B = F(S^*)$ case, the stability of the physically realistic critical points are determined by the sign of $F'(S^*)$. When a bacterial feed is included it is the sign of $R'(S^*)$ that determines the stability.

Starting from the smallest S^* value, the corresponding CPs, (S^*, X^*) , are again physically realistic, asymptotically stable, critical points for the odd numbered points, and unstable saddle points for the even numbered points.

Discussion

We have presented a straightforward analysis of the stability of a simple bioreactor with a general growth rate function, $F(S)$, with and without a bacterial feed, and death rate. Although a number of authors have considered generalised growth rate functions, $F(S)$, few have included both an influent bacteria and a death rate.

As we have seen the death rate, $K_d \geq 0$, which is generally fixed, can be added to the dilution rate, D (which can be adjusted by an operator) to get the constant value $B = D + K_d$. Adding a death rate simply restricts the range of values B might take from $(0, \infty)$ to (K_d, ∞) , while leaving the $R(S)$ and $F(S)$ functions unchanged. A particular consequence of adding a death rate is to increase smallest value the output substrate concentration, S^* , can obtain. We remind the reader that small values of D correspond to long residence times.

As we saw above if the reactor has influent bacteria then it can never washout, although it is possible to get very close to washout, which from a practical point of view is as unproductive as washout but without the need to restart the reactor. In addition since the CPs are all classified as AS, unstable saddle points, or saddle node bifurcation points, oscillations will not occur.

The number of CPs is determined by the values of B , S_{in} , X_{in} , and the function $F(S)$. For the special case of a monotonically increasing function $F(S)$, for any combination of $S_{\text{in}} > 0$ and $X_{\text{in}} > 0$, the reactor will have a unique CP, which is AS. To see this, note from (9) that if, $F'(S) > 0$ for all S , then $R'(S) > 0$ for all S , and the reactor will have a unique point of intersection of $R(S)$ and B . This behaviour was noted by Sidhu et al. [7] for a cascade of reactors with the Monod growth function, where all the reactors past the first has a unique AS CP, given that the first reactor had zero bacterial feed and was not washed out.

The converse of this however is not true. It is possible, for example, for $R(S)$ to be monotonically increasing, for some combinations of $S_{\text{in}} > 0$ and $X_{\text{in}} > 0$, when $F(S)$ is not monotonically increasing. The Haldane growth function displays this property.

The stability of the critical points (S^*, X^*) always have the same alternating pattern of stable and unstable CPs, the odd numbered (starting from the smallest S^*) correspond to AS CPs, while the even numbered correspond to unstable saddle points. This pattern holds for both reactors with a bacterial inflow, see Fig. 2, and without, see Fig. 3. This pattern of stability has been noted by other authors [3, 5].

The bifurcation points S^* values will occur when $R'(S^*) = 0$, or in the case where $X_{\text{in}} = 0$, when $F'(S^*) = 0$, and at $S^* = X_{\text{in}}$.

When considering a cascade of reactors, where the output of one reactor becomes the input of the next, S_{in} and X_{in} will be different for each reactor in the cascade (as substrate is processed) hence the $R(S)$ functions will be different for each reactor in the cascade. As a consequence even if the residence times for the reactors in the cascade are all the same, that is B is the same for all the reactors, the number of points of intersection can differ for each of the reactors in the cascade. An example of this, using the Haldane growth rate function, is given in Dramé et al. [5].

References

1. Ajbar A., K. Alhumaizi (2011). Dynamics of the Chemostat: A Bifurcation Theory Approach, Chapman and Hall/CRC.
2. Bertucco A., P. Volpe, H. E. Klei, T. F. Andreson, D. W. Sundstrom (1990). The Stability of Activated Sludge Reactors with Substrate Inhibition Kinetics and Solid Recycle, Water Research, 24(2), 169-176.
3. Calderón-Soto L. F., E. J. Herrera-López, G. Lara-Cisneros, R. Femat (2019). On Unified Stability for a Class of Chemostat Model with Generic Growth Rate Functions: Maximum Yield as Control Goal, Journal of Process Control, 77, 61-75.
4. Dochain D., P. A. Vanrolleghem (2001). Dynamical Modelling & Estimation in Wastewater Treatment Processes, IWA Publishing London.
5. Dramé A. K., J. Harmand, A. Rapaport, C. Lobry (2006). Multiple Steady State Profiles in Interconnected Biological Systems, Mathematical and Computer Modelling of Dynamical Systems, 12(5), 379-393.
6. Lara-Cisneros G., R. Femat, E. Pérez (2012). On Dynamical Behaviour of Two-dimensional Biological Reactors, International Journal of Systems Science, 43(3), 526-534.
7. Sidhu H. S., M. I. Nelson, E. Balakrishnan (2015). An Analysis of a Standard Reactor Cascade and a Step-feed Reactor Cascade for Biological Processes Described by Monod Kinetics, Chemical Product and Process Modeling, 10(1), 27-37.
8. Simeonov I., S. Diop (2010). The Stability of Activated Sludge Reactors with Substrate Inhibition Kinetics and Solid Recycle, International Journal Bioautomation, 14(1), 37-48.
9. Smith H. L., P. Waltman (1995). The Theory of the Chemostat: Dynamics of Microbial Competition, Cambridge University Press.

Andrew J. Stacey, Ph.D.

E-mail: andrew.stacey@rmit.edu.au



Andrew Stacey received his B.Sc. (Hons.) Degree in Theoretical Physics from Adelaide University, M.Sc. Degree in Applied Statistics from Macquarie University and Ph.D. Degree in Theoretical Physics from the University of New South Wales. His research interests include nonlinear systems of differential equations and perturbation analysis of biological and engineering systems.

John J. Shepherd, Ph.D.E-mail: john.shepherd@rmit.edu.au

John Shepherd received his B.Sc. Degree, Grad. Dip. Ed., M.Sc. Degree in Mathematics and Ph.D. Degree in Mathematics from the University of Melbourne. His research interests include applied differential equations, perturbation methods, nonlinear problems arising in engineering and science.



© 2022 by the authors. Licensee Institute of Biophysics and Biomedical Engineering, Bulgarian Academy of Sciences. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<http://creativecommons.org/licenses/by/4.0/>).