# Improved Speed of InterCriteria Analysis

**Alexander Marazov[1]*, Anthony Shannon[2]**

[1] *Institute of Biophysics and Biomedical Engineering*
*Bulgarian Academy of Sciences*
*Acad. Georgi Bonchev Str. Bl. 105.*
*Sofia 1113 Bulgaria*
*E-mail: alexander@biomed.bas.bg*

[2] *Warrane College University of New South Wale*
*Kensington NSW 2033 Australia*
*E-mail: tshannon@warrane.unsw.edu.au*

* *Corresponding author*

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***Abstract:*** *We will show that the computation of the intercriteria counters can be done in $O(n \log n)$ time (quasi-linear complexity). Up to this point, all implementations have used $O(n^2)$ operations, which does not allow processing of data over hundreds of thousands.*

***Keywords:*** *InterCriteria analysis, Quasi-linear complexity.*

## Introduction

The current known software implementations of Intercriteria Analysis [1] have used $O(n^2)$ operations [3, 5–7]. This complexity makes processing data over hundreds of thousands prohibitively slow. We will show that the computation of the intercriteria counters defined in [1] can be done in $O(n \log n)$ time (quasi-linear complexity).

## Notation

All the vectors we will consider later in this work are $n$-dimensional, the elements of which we can order partially with a relation "$\leq$". The elements of the vectors at different indices need not belong to the same set, but we will still write "$\leq$" for each partially ordered set.

**Definition 1.** Let $k$ and $l$ be two $n$-dimensional vectors. We say that indices $i, j$, $i < j$ are in **disagreement** [1], if and only if

$$k_i \leq k_j \ \wedge \ l_i > l_j \ \vee \ k_i > k_j \ \wedge \ l_i \leq l_j$$

**Definition 2.** We will denote by count_disagreements$(k, l)$ the number of disagreements between $k$ and $l$. There are $n(n-1)/2$ such combinations of indices $i, j$, $i < j$, which can be trivially traversed in $O(n^2)$.

We will show that the number of disagreements between $k$ and $l$, count_disagreements$(k, l)$, can be computed in $O(n \log n)$.

**Definition 3.** We say that indices $i, j$, $i < j$ are in **equality (indeterminacy)** [1] if and only if

$$k_i = k_j \ \wedge \ l_i = l_j$$

The equality can be defined in the natural way:

$$k_i = k_j \iff k_i \leq k_j \ \wedge \ k_j \leq k_i$$

**Definition 4.** We say that indices $i, j$, $i < j$ are in **agreement** [1], if and only if neither are met the definition of disagreement nor of indeterminacy.

**Definition 5. Inversion** in vector $v$ we call combinations of indices $i, j$, $i < j$ for which $v_i > v_j$ [8].

The inversions of a vector can be computed in $O(n \log n)$, as shown in [8]. The algorithm is based on a modification of *Merge Sort*, which is of complexity $O(n \log n)$ [8].

For vectors that allow equalities between elements, we will introduce the following convenient notation.

**Definition 6.** We call $\hat{v}$ the **enumerated vector** of the vector $v$[1] and we define its elements with the ordered pairs:

$$\hat{v}_i = (v_i, i)$$

We extend the order of the elements of $v$ to a lexicographic order of enumerated vectors $\hat{v}$. Since the indices in the second component of an enumerated vector are unique, equalities of enumerated vectors are not possible, even if there are equalities in the initial vector.

**Definition 7.** We will denote the number of inversions in a vector $v$ by $\mathrm{count\_inversions}(v)$.

**Definition 8.** Let $k$ and $l$ be two vectors. Let us introduce the notation $\mathrm{sort}_k(l)$ which sorts the elements of $l$ by the order of $k$.

Let us write the above definition in terms of a sorting permutation. Let us look at the permutation $\sigma$ which sorts $k$. For indices $i, j$, $i < j \Leftrightarrow k_{\sigma(i)} \leq k_{\sigma(j)}$. Then for the $i$-th element we have

$$\mathrm{sort}_k(l)_i = l_{\sigma(i)}$$

Sorting is an operation that can be performed in $O(n \log n)$ operations [8].

**Theorem 1.**

$$\mathrm{count\_disagreements}(k,l) = \mathrm{count\_inversions}(\mathrm{sort}_{\hat{k}}(\hat{l}))$$

*Proof.* We will first show that on any inversion in $\mathrm{sort}_{\hat{k}}(\hat{l}))$ there is a corresponding disagreement between $k$ and $l$. Next, we will show that for each disagreement between $k$ and $l$, there is an inversion in $\mathrm{sort}_{\hat{k}}(\hat{l})$. This will establish the existence of a bijection, proving the desired equality.

For brevity, let us denote $l^k = \mathrm{sort}_{\hat{k}}(\hat{l})$. By definition, $l^k_i = \hat{l}_{\sigma(i)}$, where $\sigma$ is the permutation, which sorts $\hat{k}$.

($\Leftarrow$) Let us fix an inversion in $l^k$ with indices $i'$, $j'$ for which $i' < j'$ and $l^k_{i'} > l^k_{j'}$.

Let $i$, $j$ be the images of $i'$, $j'$ in $\sigma$: $i = \sigma(i')$, $j = \sigma(j')$. Then we have

$$\hat{l}_i = \hat{l}_{\sigma(i')} = l^k_{i'} > l^k_{j'} = \hat{l}_{\sigma(j')} = \hat{l}_j$$

[1] Similar to the function in Python enumerate, which gives a sequence $(i, v_i)$ from $v$.

But $\hat{l}_i > \hat{l}_j$ is only possible when

$$l_i \geq l_j$$

Let us look into the two cases for $i$ and $j$: $i < j$ or $i > j$.

Let the first case $i < j$ hold. From the definition of $\sigma$ and $\sigma(i') < \sigma(j')$ it follows that $k_i \leq k_j$. From $\hat{l}_i > \hat{l}_j$ and $i < j$ it follows that $l_i > l_j$. The equality $l_i = l_j$ is impossible because it would mean that $i > j$. But from $k_i \leq k_j$ and $l_i > l_j$ we get a disagreement.

Let the second case $i > j$ hold. From $\hat{l}_i > \hat{l}_j$ and $i > j$ it follows that $l_i \geq l_j$. A tie in this case is possible. From the definition of $\sigma$ and $i' < j'$ it follows that $\hat{k}_i \leq \hat{k}_j$. It's here essential that $\sigma$ sorts $\hat{k}$ to exclude the equality $k_i = k_j$. If we assume that the equality is fulfilled, this will mean that $i < j$, which contradicts the case under consideration. It remains $k_i > k_j$ $l_i \geq l_j$, which means that $j$ and $i$ with $j < i$ are disagreement in $k$ and $l$.

With this, the direction ( $\Leftarrow$ ) is proved.

($\Rightarrow$) Let disagreement be fixed between $k$ and $l$ with indices $i$ and $j$, $i < j$

$$k_i \leq k_j \wedge l_i > l_j \vee k_i > k_j \wedge l_i \leq l_j$$

Let $i'$ and $j'$ be the primes of $i$ and $j$ in $\sigma$: $i = \sigma(i')$, $j = \sigma(j')$.

$$l^k_{i'} = \hat{l}_{\sigma(i')} = \hat{l}_i$$

$$l^k_{j'} = \hat{l}_{\sigma(j')} = \hat{l}_j$$

We have two possible cases:

The first of them is

$$k_i \leq k_j \wedge l_i > l_j$$

$l_i > l_j$ leads to $l^k_{i'} > l^k_{j'}$.

From $k_{\sigma(i')} \leq k_{\sigma(j')}$ and the definition of $\sigma$ follows that $i' < j'$. The latter means that $i'$, $j'$ is an inversion in $l^k$.

The second possible case is

$$k_i > k_j \wedge l_i \leq l_j$$

From $k_{\sigma(i')} > k_{\sigma(j')}$ and the definition of $\sigma$ it follows that $i' > j'$. The inequality $l_i \leq l_j$ implies $l^k_{i'} \leq l^k_{j'}$. Let us look at when equality is reached. $l^k_{i'} = l^k_{j'} \Leftrightarrow \hat{l}_i = \hat{l}_j$. By the definition of $\hat{l}$, for this to hold, $i = j$, so with the vector $\hat{l}$ listed. This leads to a contradiction. Therefore there can be no equality, which makes $j'$, $i'$ inversion in $l^k$.

With this, the direction ($\Rightarrow$) is proved.

$\square$

## Calculation of intercriteria counters

Let us look at the definitions of intercriteria counters in [1] for two criteria $k$ and $l$. We will denote the object at index $k$ with the index of the vector for brevity. Using the standard notation from [1], $S^{\mu}_{kl}$ is the number of agreements, $S^{\nu}_{kl}$ is the number of disagreements, $S^{\pi}_{kl}$ is the number of equalities.

Using Theorem 1, we proved that $S^{\nu}_{kl}$ can be computed in $O(n \log n)$.

Let us see how to calculate $S^{\pi}_{kl}$ in $O(n \log n)$. We form a vector $z$ with elements $z_i = (k_i, l_i)$. We can sort $z$ with the natural lexicographic ordering in $O(n \log n)$ operations. To determine the number of elements that are equal to each other, we traverse the sorted vector $z$ once. Let $n_a$ be the number of elements equal to a given value $z_a$. We can calculate with the formula $n_a(n_a - 1)/2$ the number of combinations of elements. Finally, we sum these values for each different value of $z_a$ to obtain $S^{\pi}_{kl}$.

We calculate the number of agreements by subtracting the other two counters from the number of combinations:

$$S^{\mu}_{kl} = \frac{n(n-1)}{2} - S^{\pi}_{kl} - S^{\nu}_{kl}$$

## Open problems

The computations of the degree of disagreement of the InterCriteria Analysis [1] share similarity to the Kendall metric $\tau$ [9]. It is known that the Kendall metric $\tau$ can be reduced to counting inversions [4]. As we have shown, we can reduce the computation of count_disagreements to counting inversions. A result which allows even counting inversions with $O(n\sqrt{\log n})$ is presented of Chan and Patrasku [2].

Since for the equality counter calculation $S^{\pi}_{kl}$ we did not use inversions, the question, whether the intercriteria counters can be calculated for $O(n\sqrt{\log n})$, remains open.

## Conclusion

The existing software implementations of InterCriteria Analysis have been observed to employ $O(n^2)$ operations for their processing tasks. This quadratic complexity presents a significant limitation, particularly when handling large datasets, rendering the processing of hundreds of thousands of data points impractically slow.

Our study aimed to demonstrate a more efficient approach. Specifically, we proposed that the computation of the InterCriteria counters can be accomplished in $O(n \log n)$ time complexity. This improvement signifies a shift towards quasi-linear complexity, offering a more scalable and expedient solution for data processing tasks.

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**Alexander Marazov, Ph.D. Student**
E-mail: alexander@biomed.bas.bg


Alexander Marazov received his B.Sc. Degree in Applied Mathematics in 2009 and M.Sc. Degree in Mathematical Modeling and Numerical Methods in 2013, both from Sofia University "St. Kliment Ohridski", Bulgaria. He is currently a Ph.D. student at the Institute of Biophysics and Biomedical Engineering, Bulgarian Academy of Science.


**Emeritus Professor Anthony Shannon, Ph.D., Ed.D., D.Sc.**
E-mail: tshannon@gmail.com


Professor A. G. (Tony) Shannon AM is a Member of the Order of Australia and an Emeritus Professor of the University of Technology, Sydney. He has just completed service as the Deputy Chancellor of the University of Notre Dame Australia, and he is currently a Director of Academic Studies at the Australian Institute of Music. He holds the Doctoral Degrees of Ph.D., Ed.D. and D.Sc. He is co-author of numerous publications in medicine, mathematics and education. He enjoys reading, walking, theatre, number theory, watching rugby, and thoroughbred racing.