INT. J. BIOAUTOMATION, 2024, **28**(4), 267-271 doi: 10.7546/ijba.2024.28.4.001039

Book Review

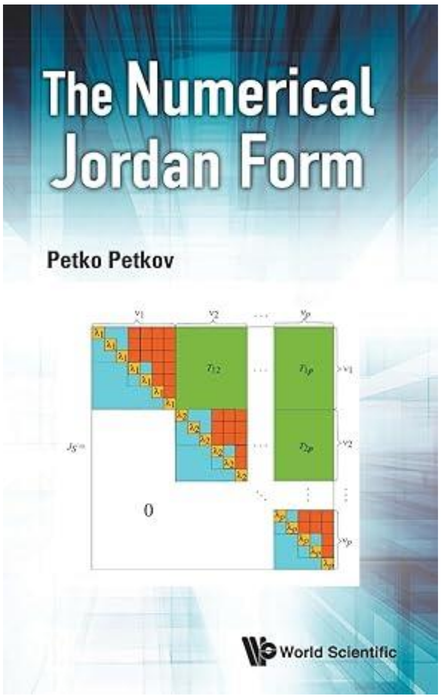
"Pure" or "Numerical" Jordan Form?

Ivan Popchev

Bulgarian Academy of Sciences
E-mail: *ivan.popchev@iict.bas.bg*

Received: December 15, 2024 **Accepted: December 20, 2024**

Published: December 31, 2024



World Scientific
Singapore, 2024
ISBN: 978-981-12-8644-5
630 pages
https://doi.org/10.1142/13682

The Jordan canonical form is a powerful tool for solving several problems arising in the theory of matrices and matrix computations since it contains the whole information about the algebraic structure of the underlying linear transformation and the corresponding matrix [1-4]. With the aid of the Jordan form, important theoretical and applied problems are solved including the stability of linear time-invariant systems, computation of matrix functions, and solution of matrix equations. That is why the Jordan form has applications in several scientific disciplines such as oscillations theory, circuit theory and control theory, to name a few. Unfortunately, there is a widespread misconception that the Jordan form is not suitable for numerical computations. This opinion is based on the fact that the "pure" or "theoretical" Jordan form is not a continuous function of the matrix elements and thus its computation is an ill-posed problem. As opposite to the theoretical case, the numerical Jordan form is defined by regularization of an ill-conditioned eigenvalue problem and is not sensitive to small changes of the matrix elements. Hence it may be computed stably in most cases. As a "by-product" of the reduction of a matrix into its numerical Jordan form, one finds at an intermediate step the Weyr canonical form of the matrix. Also, as a result of the regularization procedure, the transformation to numerical Jordan form is always better conditioned than the eigenvector matrix computed by the available eigenvalue routines which allow computing reliably matrix functions. Besides, the numerical methods for finding the Jordan form have great pedagogical value since they make use of the best algorithms for matrix computations.

The purpose of the book under review is to describe in detail the numerical difficulties in determining the Jordan form of a matrix and to present the available state-of-the art algorithms for reducing a square matrix into numerical Jordan form. In contrast to the “pure” Jordan form whose computation is ill posed, the numerical Jordan form preserves its structure under small perturbations of the matrix elements, making its determination a well-posed computational problem. If the structure is well conditioned, it can be determined reliably in the presence of uncertainties and rounding errors.

The book focuses on the algorithmic and computational aspects of finding the Jordan form of a matrix. It offers a comprehensive presentation of the famous algorithm of Kublanovskaya-Ruhe-Kågström [5-7], designed to transform a complex matrix into numerical Jordan form. Additionally, it delves into the practical application of this form in solving significant problems, such as estimating eigenvalue sensitivity and computing the matrix exponential. A particular emphasis is placed on the Jordan-Schur form of a matrix, which, in the author’s perspective, remains underutilized in the field of matrix computations. Since the mathematical objects under consideration can be sensitive to changes in the elements of the given matrix, the book extensively covers perturbation analysis pertaining to eigenvalues, invariant subspaces, Schur form, and Jordan form. A large number of numerical examples and figures are included to clarify mathematical concepts.

Here is a brief synopsis of the contents. The book consists of nine chapters and an Appendix. The purpose of the **First chapter** is to review the necessary concepts related to the numerical solution of matrix problems that will be needed in the remaining part of the book. First, a brief analysis is done of the errors arising in the representation of numbers and in performing the basic arithmetic operations in the IEEE floating point arithmetic. Error bounds for the most frequently used matrix operations are derived in detail. Next, the author considers the conditioning of computational problems and the numerical stability of algorithms for their solution. He presents the construction and analyzes the errors associated with computing of the elementary matrix transformations which are the building blocks in developing the numerical methods for matrix computations. The implementation of these transformations is illustrated by the computation of the QR decomposition of a matrix. Finally, he defines the important notion of numerical rank and show how to compute the fundamental subspaces related to a given matrix.

In the **Second chapter** the author gives the basic facts relevant to the eigenvalue problem which are used in the remainder of the book. He considers the properties of the eigenvalues and eigenspaces, the emphasis being put on Schur, Jordan and Weyr [8] canonical forms. Several facts are given without proofs which can be found in the sources referred to at the end of each section.

In the **Third chapter** is presented a study on the sensitivity of the eigenvalues of a matrix to small changes (perturbations) of its elements. Since the multiple eigenvalues associated with nonlinear elementary divisors are usually very sensitive to perturbations of the matrix, the eigenvalue sensitivity is closely related to the problem of determining the Jordan structure of the matrix. The investigation of the matrix eigenvalue sensitivity is a subject to the *eigenvalue perturbation theory* which is an important part of the matrix analysis. Also, in a separate section is considered the sensitivity of the generalized eigenvalue problem, which arise in connection with the determining of Weierstrass and Kronecker canonical form.

The transformation into upper triangular (Schur) form and determination of the eigenvalues is the first step in the reduction of a square matrix into Jordan canonical form. In the **Fourth chapter** the author discusses in brief some basic properties of the QR algorithm used in the reduction into Schur form and computing approximations of the eigenvalues of a matrix. He also presents the solution of the Sylvester matrix equation and its application to the computation of spectral projections and estimating the sensitivity of invariant subspaces.

In the **Fifth chapter** the author presents the notion of numerical Jordan form of a matrix. The presentation is based on the geometry of matrix space divided into orbits and bundles, according to the famous theory of matrices depending on parameters, developed by Arnold [9]. In contrast to the "theoretical" Jordan form, the numerical Jordan form preserves its structure when the matrix is subject to small perturbations. This makes the numerical Jordan form a computational object that can be determined reliably in the presence of rounding errors. To find the Jordan structure, it is necessary to know the geometry of the space of matrices and the stratification of the near Jordan forms in this space. This important subject is discussed in simplified form in the corresponding sections of this chapter. In Sect. 5.4 is described the determination of the Jordan form as a solution of an ill posed numerical problem. A rigorous definition of the numerical Jordan form is given in Sect. 5.5 and a practical definition, appropriate for computations, is presented in Sect. 5.6.

In the **Sixth chapter** are presented numerical algorithms intended to obtain the Jordan-Schur form of a square matrix. The Jordan-Schur form is an intermediate step in determining the Weyr and Jordan forms and has independent applications. The chapter begins with a short review of the available algorithms for determining the Jordan structure of a matrix. After a brief description of the staircase form and the Jordan-Schur form, the author discusses three basic algorithms whose sequential application allows reducing the Schur form of a matrix into the Jordan-Schur form. The numerical properties of these algorithms are evaluated and some information about their implementation using LAPACK and MATLAB is given. The numerical computation of the Jordan-Schur form is in the basis of the algorithm for finding the Jordan form, considered in the next chapter.

The **Seventh chapter** presents the full algorithm of the reduction of a matrix into numerical Jordan form. An intermediate step of the reduction is the determining of the Weyr characteristics and the Weyr form of the matrix.

In the **Eighth chapter** the author presents a case study devoted to the application of the block-diagonal and the numerical Jordan form of a matrix to the computing of eigenvalue sensitivity estimates. In order to obtain tighter bounds, the estimates obtained by using the Henrichi theorem are optimized. It is shown that the implementation of the Jordan form leads to minimum sensitivity bounds as a result of using diagonal blocks of minimum size.

The computation of the matrix exponential of a real or complex square matrix is an important task in the solution of differential equations, obtaining the time response and discretization of control systems, computing the integral of the exponential and several other problems arising in applied mathematics. In the case study, presented in **Ninth chapter**, the author considers some methods for numerical computation of the matrix exponential putting the emphasis on the using of Jordan and Jordan-Schur form. After an introduction into the properties of the exponential, the author presents a brief review of some numerical methods intended for computing the matrix exponential. In detail are presented four methods for evaluation of the exponential based on the numerical Jordan form, block-diagonal staircase form, Schur-Parlett and Schur-Fréchet methods.

In the **Appendix** are given the most important definitions and results about matrices, linear spaces and linear transformations which are used throughout the book. It is assumed that the reader has some knowledge about matrices and operations on matrices.

In total, the book presents 40 algorithms for computing the Jordan, Weyr, and Jordan-Schur forms of matrices, along with various mathematical objects related to eigenvalue computations.

Given the increased complexity in determining these forms for real matrices, the algorithms are derived and described within the complex domain. The book employs a MATLAB-like notation, and it comes with a collection of MATLAB m-files implementing these algorithms.

Regarding the numerical experiments featured in the book, most of the examples have been intentionally designed in a way that their analytical solutions are known and can be computed with high precision. This deliberate design aids in identifying the reasons behind rounding errors in intermediate and final results, as well as pinpointing potential sources of numerical instability. The author strongly encourages readers to perform these experiments themselves, as they offer valuable insights into the numerical behavior of the methods under investigation. Engaging in these experiments also forms an integral part of comprehending the book's content and gaining practical computational experience.

The text may serve as a comprehensive, self-contained reference for the numerical determination of the Jordan form of a matrix and its practical applications. It covers essential material from matrix analysis, numerical computations, and linear algebra. The computational algorithms for finding the Jordan, Weyr, and Jordan-Schur forms are introduced step by step, with detailed examples illustrating the concepts. A prerequisite for readers is some familiarity with matrices and basic knowledge of analysis. The intended audience comprises specialists from various scientific disciplines who engage in matrix computations but may not be experts in numerical analysis. Consequently, the author has minimized theoretical rigor, avoiding detailed proofs of certain results. Instead, readers are directed to relevant literature for in-depth proof of mathematical facts. This book can also benefit advanced master's students studying topics like Matrix Computations, Numerical Analysis, Control Theory, and related fields.

It's worth noting that this book serves as an introductory resource to the numerical determination of the Jordan form. A deeper exploration of related issues, which involve multiple mathematical disciplines like algebraic geometry, differential geometry, topology, singularity theory, and bifurcation theory, requires more extensive knowledge. The book references more than 400 literature sources for those interested in further study.

Further results of the author and his co-workers in the area of numerical computations are given in [10, 11].

The World Scientific Publisher should be congratulated with the publication of this very useful and informative book which is the first one devoted entirely on the numerical issues related to the computation of the Jordan form.

References

1. Jordan C. (1870). Traité des Substitutions et des Équations Algébriques, Gauthire-Villars, Paris.
2. Gantmacher F. R. (1959). Theory of Matrices, Chelsea, New York, Reprinted by the AMS, 2000.
3. Gohberg I., P. Lancaster, L. Rodman (2006). Invariant Subspaces of Matrices with Applications. Classics in Applied Mathematics, Society for Industrial and Applied Mathematics, Philadelphia, PA, http://dx.doi.org/10.1137/1.9780898719093.
4. Horn R. A., C. R. Johnson (2013). Matrix Analysis, 2nd Ed., Cambridge University Press, Cambridge, UK.
5. Kublanovskaya V. N. (1966). On a Method of Solving the Complete Eigenvalue Problem for a Degenerate Matrix, USSR Computational Mathematics and Mathematical Physics, 6, 1-14, http://dx.doi.org/10.1016/0041-5553(66)90001-2.
6. Ruhe A. (1970). An Algorithm for Numerical Determination of the Structure of a General Matrix, BIT Numerical Mathematics, 10, 196-216, http://dx.doi.org/10.1007/BF01936867.
7. Kågström B., A. Ruhe (1980). An Algorithm for Numerical Computation of the Jordan Normal Form of a Complex Matrix, ACM Trans Math Software, 6, 398-419, http://dx.doi.org/10.1145/355900.355912.
8. Weyr E. (1890). Zur Theorie der bilinearen Formen, Monatshefte für Mathematik und Physik, 1, 163-236, http://dx.doi.org/10.1007/BF01692475.
9. Arnold V. I. (1971). On Matrices Depending on Parameters, Russian Math Surveys, 26, 29-43, http://dx.doi.org/10.1070/rm1971v026n02abeh003827.
10. Angelova V., P. H. Petkov (2024). Componentwise Perturbation Analysis of the Singular Value Decomposition of a Matrix, Applied Sciences, 14, 1417, http://dx.doi.org/10.3390/app14041417.
11. Angelova V., M. Konstantinov, P. Petkov (2024). Asymptotic and Probabilistic Perturbation Analysis of Controllable Subspaces, Computation, 12, 236, http://dx.doi.org/10.3390/computation12120236.