

# Twenty Fourth International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets

## Intuitionistic Fuzzy Nomograms

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# Introduction

In clinical practice, nomograms present relationships between patient characteristics, test results, and diagnostic or prognostic outcomes as graphical scales. Their mathematical origins lie in nineteenth-century nomography, closely associated with the French engineer Maurice d'Ocagne, whose *Traité de Nomographie* systematized methods of graphical calculation. Contemporary clinical nomograms can be constructed from mathematical relationships, observed data, or fitted statistical models. In the most common point nomogram, the user reads the contribution of each predictor, sums the points, and obtains an outcome estimate from the total. The graphical form makes the result accessible, but its interpretation depends on the underlying model and on the outcome being estimated.

A practical difficulty arises when the value of a required predictor is unknown. The known predictors provide only part of the total score, although the nomogram specifies the points assigned to each admissible value of the missing predictor. For an additive point nomogram, these admissible values can be used to calculate lower and upper bounds on the total, and if a mapping from total points to outcomes has been specified, the attainable scores also determine bounds on the resulting estimate. This interval describes the consequences of incomplete input information; it is not a confidence interval or a probability distribution for the missing value.

Uncertainty has previously been incorporated into nomogram displays, for example through confidence intervals for estimated model contributions and probabilities (Možina, M., Demšar, J., Kattan, M.W., Zupan, B. (2004): Nomograms for visualization of naive Bayesian classifier. In: Boulicaut, J.-F., Esposito, F., Giannotti, F., Pedreschi, D. (eds.) PKDD 2004. LNCS (LNAI), vol. 3202, pp. 337–348. Springer, Berlin, Heidelberg).

Interval methods have also been applied to medical decision support when input data are incomplete (Wójtowicz, A., Żywica, P., Stachowiak, A., Dyczkowski, K. (2016): Solving the problem of incomplete data in medical diagnosis via interval modeling. Applied Soft Computing **47**, 424–437).

A different question arises when the predictor values are known, but individuals assigned to the same subgroup have different recorded outcomes. Representing that variation is the focus of the present work. We use intuitionistic fuzzy pairs, an ordered couple of non-negative numbers which sum is less than or equal to one, to place subgroup outcome ranges on a common graphical scale and examine how the representation changes as subgroups are refined.

# Intuitionistic fuzzy nomograms for subgroup outcome ranges

This section develops an intuitionistic fuzzy nomogram for the variation in recorded outcomes within groups. We first place each group's observed range on a common scale and express it as an intuitionistic fuzzy pair. We then describe how the representation changes when observations are added or groups are refined, and how modal values and medians support the proposed outcome assessments.

# Observed outcome ranges and their graphical representation

Let  $\mathcal{P}$  be a set of  $p$  observations. A specified criterion  $\mathcal{C}$  assigns each observation to exactly one of  $f$  groups  $F_1, \dots, F_f$ . Denote by  $\mathcal{P}_i = \{P_{i,1}, \dots, P_{i,q_i}\}$  the non-empty set of observations in group  $F_i$ . The sets  $\mathcal{P}_i$  form a partition of  $\mathcal{P}$ ; hence

$$\mathcal{P} = \bigcup_{i=1}^f \mathcal{P}_i, \quad p = \sum_{i=1}^f q_i.$$

The construction also allows for a single group ( $f = 1$ ), which may contain any number of observations.

Let  $t_{i,j}$  be the recorded outcome for observation  $P_{i,j}$ , where  $1 \leq i \leq f$  and  $1 \leq j \leq q_i$ . When the outcome is a duration, all  $t_{i,j}$  must refer to the same starting point and use the same unit of time. Choose reference values  $\tau_0$  and  $\sigma_0$  such that

$$\tau_0 < \min_{\substack{1 \leq i \leq f \\ 1 \leq j \leq q_i}} t_{i,j} \leq \max_{\substack{1 \leq i \leq f \\ 1 \leq j \leq q_i}} t_{i,j} < \sigma_0. \quad (1)$$

These values define a common reference interval  $[\tau_0, \sigma_0]$  for all groups. They need not be interpreted as theoretical limits of the outcome.

For each group, define its observed lower and upper bounds by

$$\tau_i = \min_{1 \leq j \leq q_i} t_{i,j}, \quad \sigma_i = \max_{1 \leq j \leq q_i} t_{i,j}. \quad (2)$$

Thus,  $\tau_0 < \tau_i \leq \sigma_i < \sigma_0$  for every  $i$ . Figure 1 represents each group on its own horizontal line, using marked points for its observed bounds and a common horizontal scale for the outcome. The dashed vertical lines indicate the positions of the reference bounds and the two bounds shown for  $F_i$ .

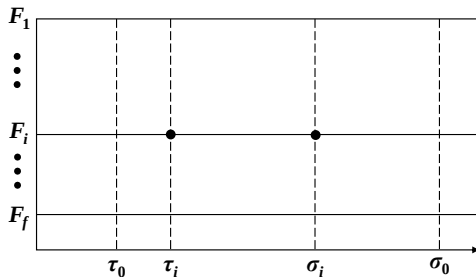


**Figure 1:** Graphical representation of the observed outcome interval  $[\tau_i, \sigma_i]$  for group  $F_i$  within the common reference interval  $[\tau_0, \sigma_0]$ .

The intervals of different groups may overlap: membership in  $F_i$  is determined by criterion  $C$ , not by an assumed ordering of the observed outcomes. The graphical construction records these intervals on a shared scale. Its intuitionistic fuzzy interpretation is introduced next.

# From subgroup intervals to intuitionistic fuzzy pairs

The common reference interval  $[\tau_0, \sigma_0]$  provides a scale on which the observed bounds of every group can be expressed. Let  $D = \sigma_0 - \tau_0 > 0$ . For each group  $F_i$ ,  $1 \leq i \leq f$ , define

$$\mu_i = \frac{\tau_i - \tau_0}{D}, \quad \nu_i = \frac{\sigma_0 - \sigma_i}{D}. \quad (3)$$

The interval  $[\tau_i, \sigma_i]$  is thus represented by the intuitionistic fuzzy pair  $\langle \mu_i, \nu_i \rangle$ . The same construction applies to every group; it does not require an ordering of the groups or non-overlapping outcome intervals.

By (1) and (2), both components lie in  $[0, 1]$ . Moreover,

$$\mu_i + \nu_i = 1 - \frac{\sigma_i - \tau_i}{D} \leq 1. \quad (4)$$

Hence,  $\langle \mu_i, \nu_i \rangle$  is a valid intuitionistic fuzzy pair. Its residual degree of indeterminacy is

$$\pi_i = 1 - \mu_i - \nu_i = \frac{\sigma_i - \tau_i}{D}. \quad (5)$$

In this construction,  $\pi_i$  is the width of the group's observed outcome interval relative to the reference interval. If all recorded outcomes in a group coincide, then  $\pi_i = 0$ . This does not imply that the outcome of a future observation is known with certainty.

For a new case assigned to  $F_i$ , the pair summarizes the outcomes recorded for that group; it is not, by itself, a probability of a particular outcome or a guarantee that the new outcome will lie between  $\tau_i$  and  $\sigma_i$ . The interpretation also depends on the chosen reference interval: changing  $\tau_0$  or  $\sigma_0$  changes the numerical components, even if the observed group bounds remain the same. As additional observations become available, the group bounds and their corresponding pairs can be updated.

## Updating subgroup bounds and IF pairs

Suppose that a new observation is assigned to group  $F_i$  and has a recorded outcome  $t$ . After confirming the group assignment and the value of  $t$ , the observed bounds of that group become

$$\tau_i^{\text{new}} = \min\{\tau_i, t\}, \quad \sigma_i^{\text{new}} = \max\{\sigma_i, t\}. \quad (6)$$

If  $\tau_i \leq t \leq \sigma_i$ , neither bound changes, although the number of observations in the group increases. Otherwise, exactly one of its two bounds changes. An unexpected value may warrant checking the record or its group assignment, but overlap with the outcome interval of another group does not, by itself, constitute a classification error.

If the new outcome remains inside the common reference interval  $[\tau_0, \sigma_0]$ , the updated intuitionistic fuzzy pair for  $F_i$  is obtained by substituting  $\tau_i^{\text{new}}$  and  $\sigma_i^{\text{new}}$  into (3). The pairs for the other groups remain unchanged. If  $t$  falls outside the reference interval, new reference bounds  $\tau_0^{\text{new}} < t < \sigma_0^{\text{new}}$  must be chosen so that they also contain all earlier outcomes. All group pairs must then be recalculated against this same new interval, even when their observed bounds have not changed.

Thus, an update has two possible effects: it may alter the observed bounds of the group receiving the new observation, and it may require a change of the common reference scale. The rule used to choose or revise that scale should be reported, since pairs calculated with different reference intervals are not directly comparable.

For a time-to-event outcome, a censored follow-up time is not an observed event time and cannot be inserted into (6) as a completed duration. Such observations require separate treatment in a survival analysis.

## Recursive refinement by additional criteria

A group in the nomogram may be subdivided when another criterion is available for its observations. Let  $D$  be such a criterion for group  $F_i$ , and suppose that it partitions  $\mathcal{P}_i$  into  $g_i$  non-empty subsets  $\mathcal{P}_{i,1}, \dots, \mathcal{P}_{i,g_i}$ . These subsets are mutually disjoint and satisfy

$$\mathcal{P}_i = \bigcup_{k=1}^{g_i} \mathcal{P}_{i,k}, \quad q_i = \sum_{k=1}^{g_i} r_{i,k}, \quad r_{i,k} = |\mathcal{P}_{i,k}| \geq 1. \quad (7)$$

Each observation in  $F_i$  therefore appears in exactly one of its refined subgroups.

For each  $\mathcal{P}_{i,k}$ , let  $\tau_{i,k}$  and  $\sigma_{i,k}$  be the minimum and maximum recorded outcomes among its observations. Because the subgroup contains only observations from  $\mathcal{P}_i$ ,

$$\tau_i \leq \tau_{i,k} \leq \sigma_{i,k} \leq \sigma_i. \quad (8)$$

Using the same reference interval  $[\tau_0, \sigma_0]$  as for the parent group, the refined subgroup receives the pair

$$\left\langle \frac{\tau_{i,k} - \tau_0}{\sigma_0 - \tau_0}, \frac{\sigma_0 - \sigma_{i,k}}{\sigma_0 - \tau_0} \right\rangle. \quad (9)$$

The pair of the parent group remains available as a summary of all its observations. The subgroup pair describes the narrower set selected by criterion  $D$ ; it does not replace the parent pair for the other subgroups. Figure 2 shows the refinement of two selected groups,  $F_i$  and  $F_j$ , on the right of the original nomogram. Their subdivisions need not have the same number of branches.

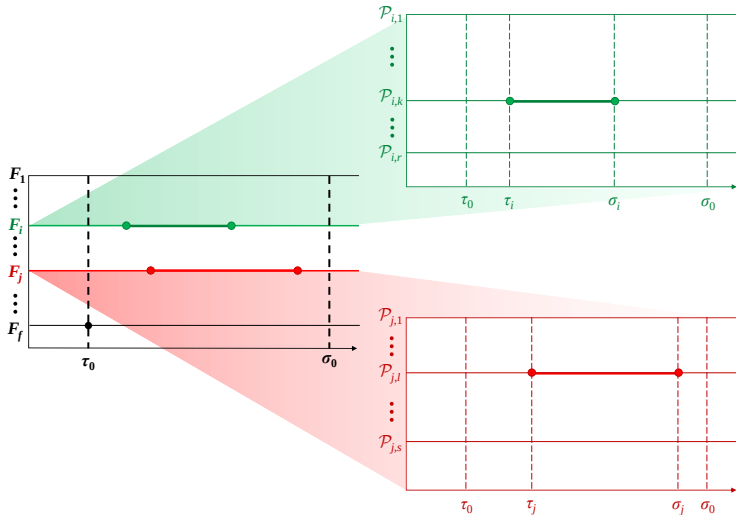


**Figure 2:** Refinement of selected groups by an additional criterion. Each right-hand branch contains a subset of the observations in its parent group; all outcome bounds are shown relative to the common reference interval  $[\tau_0, \sigma_0]$ .

The same procedure can be applied to any resulting subgroup using a further criterion. This produces a cascade of progressively more specific groups. At every level, the number of observations and the criterion defining each branch should be reported alongside its outcome bounds and intuitionistic fuzzy pair. A smaller observed interval describes the selected observations; by itself, it does not establish greater predictive accuracy for future cases.

# Modes and medians of subgroup outcomes

The bounds  $\tau_i$  and  $\sigma_i$  describe the observed range of outcomes in group  $F_i$ , but do not indicate where its observations are concentrated. To obtain additional summaries, let  $\theta_{i,1} < \theta_{i,2} < \cdots < \theta_{i,m_i}$  be the distinct recorded outcome values in that group, and let  $n_{i,\ell}$  be the number of observations with outcome  $\theta_{i,\ell}$ . Thus,

$$n_{i,\ell} \geq 1, \quad \sum_{\ell=1}^{m_i} n_{i,\ell} = q_i.$$

The values and their frequencies can be displayed as follows:

|               |                |                |          |                  |
|---------------|----------------|----------------|----------|------------------|
| Outcome value | $\theta_{i,1}$ | $\theta_{i,2}$ | $\cdots$ | $\theta_{i,m_i}$ |
| Frequency     | $n_{i,1}$      | $n_{i,2}$      | $\cdots$ | $n_{i,m_i}$      |

Define the highest observed frequency by

$$\Theta_i = \max_{1 \leq \ell \leq m_i} n_{i,\ell}. \quad (10)$$

There may be more than one outcome value with this frequency. We denote the smallest and largest of these modal values by

$$\theta_{i,\min} = \min\{\theta_{i,\ell} : n_{i,\ell} = \Theta_i\}, \theta_{i,\max} = \max\{\theta_{i,\ell} : n_{i,\ell} = \Theta_i\}. \quad (11)$$

If the mode is unique, then  $\theta_{i,\min} = \theta_{i,\max}$ . Values lying between two distinct modal values are not necessarily modal themselves.

To define the median, arrange the  $q_i$  individual recorded outcomes, including repeated values, in non-decreasing order:

$t_{i,(1)} \leq t_{i,(2)} \leq \cdots \leq t_{i,(q_i)}$ . The median outcome for group  $F_i$  is

$$\theta_{M,i} = \begin{cases} t_{i,((q_i+1)/2)}, & q_i \text{ odd,} \\ \frac{t_{i,(q_i/2)} + t_{i,(q_i/2+1)}}{2}, & q_i \text{ even.} \end{cases} \quad (12)$$

In the even case, the median need not coincide with a recorded value. The modal values and median provide reference points for the prognostic assessments introduced in the next section; they do not change the subgroup bounds or the intuitionistic fuzzy pair defined earlier.

# Pessimistic and optimistic outcome assessments

The subgroup bounds, modal values, and median can be used to describe a range of outcome scenarios. To express each scenario as an intuitionistic fuzzy pair, define, for  $\tau_0 \leq \mathbf{a} \leq \mathbf{b} \leq \sigma_0$ ,

$$\Phi(\mathbf{a}, \mathbf{b}) = \left\langle \frac{\mathbf{a} - \tau_0}{\sigma_0 - \tau_0}, \frac{\sigma_0 - \mathbf{b}}{\sigma_0 - \tau_0} \right\rangle. \quad (13)$$

Its indeterminacy degree is  $(\mathbf{b} - \mathbf{a})/(\sigma_0 - \tau_0)$ . Thus,  $\Phi(\mathbf{a}, \mathbf{b})$  represents the interval  $[\mathbf{a}, \mathbf{b}]$  on the same reference scale used for the subgroup pairs.

First, suppose that  $\theta_{i,\min} < \theta_{i,\max}$ . We choose a central representative value

$$c_i = \begin{cases} \theta_{M,i}, & \theta_{M,i} \in [\theta_{i,\min}, \theta_{i,\max}], \\ \frac{\theta_{i,\min} + \theta_{i,\max}}{2}, & \theta_{M,i} \notin [\theta_{i,\min}, \theta_{i,\max}]. \end{cases} \quad (14)$$

The five assessments are then:

| Assessment           | Representative value | IF pair                                  |
|----------------------|----------------------|------------------------------------------|
| Strongly pessimistic | $\tau_i$             | $\Phi(\tau_0, \tau_i)$                   |
| Weakly pessimistic   | $\theta_{i,\min}$    | $\Phi(\tau_i, \theta_{i,\min})$          |
| Average              | $c_i$                | $\Phi(\theta_{i,\min}, \theta_{i,\max})$ |
| Weakly optimistic    | $\theta_{i,\max}$    | $\Phi(\theta_{i,\max}, \sigma_i)$        |
| Strongly optimistic  | $\sigma_i$           | $\Phi(\sigma_i, \sigma_0)$               |

In particular, the membership component of the strongly optimistic pair is  $(\sigma_i - \tau_0)/(\sigma_0 - \tau_0)$ . The five pairs correspond to consecutive intervals from  $\tau_0$  to  $\sigma_0$ ; their membership components are non-decreasing and their non-membership components are non-increasing.

If the mode is unique, let  $\theta_{i,\min} = \theta_{i,\max} = \theta_i^*$ . The central modal interval then reduces to a single point. Following the four-assessment construction, choose an anchor  $Z_i$ : either the modal value  $\theta_i^*$  or the median  $\theta_{M,i}$ . The same choice must be used for both weak assessments:

| Assessment           | Representative value | IF pair                    |
|----------------------|----------------------|----------------------------|
| Strongly pessimistic | $\tau_i$             | $\Phi(\tau_0, \tau_i)$     |
| Weakly pessimistic   | $(\tau_i + Z_i)/2$   | $\Phi(\tau_i, Z_i)$        |
| Weakly optimistic    | $(Z_i + \sigma_i)/2$ | $\Phi(Z_i, \sigma_i)$      |
| Strongly optimistic  | $\sigma_i$           | $\Phi(\sigma_i, \sigma_0)$ |

The chosen anchor should be stated whenever these assessments are reported.

These labels describe positions relative to the observed subgroup distribution and the selected reference interval. In particular, “average” denotes the central scenario specified in (14); it does not necessarily denote the arithmetic mean. None of the pairs is, without an additional prediction model, a probability that a new case will attain the representative value.

# Scope and Potential Applications

The intuitionistic fuzzy nomogram can be used when observations can be assigned to subgroups and their outcomes measured on a common ordered scale. The grouping criteria may be categorical or may arise by partitioning a numerical characteristic into specified ranges. For each subgroup, the nomogram displays the observed outcome interval, its intuitionistic fuzzy pair, and, where useful, modal values and the median. Further criteria can be applied to selected groups, producing a graphical cascade of more specific subgroups.

In a clinical setting, this construction may help researchers explore variation in recorded outcomes, compare patient subgroups, and inspect how their descriptions change as new observations are added. The same principle can be applied outside medicine whenever subgroup outcome ranges are of interest. Its use requires comparable outcome measurements, an explicitly chosen reference interval, and sufficient observations to make the displayed subgroup summaries informative. For time-to-event outcomes, censored follow-up requires separate treatment; the nomogram alone does not estimate individual survival probabilities.

An interactive research prototype implementing the intuitionistic fuzzy nomogram and its recursive subgroup refinement is available at <https://biomed.bas.bg/en/projects/kp-06-n43-7/software>. The prototype illustrates the proposed method; its use for clinical decision support would require validation in the intended patient population.

# Discussion and Conclusion

The intuitionistic fuzzy nomogram proposed in this paper represents the range of outcomes recorded within each subgroup on a common graphical scale. The lower and upper bounds determine an intuitionistic fuzzy pair, while its residual indeterminacy reflects the relative width of the observed range. The nomogram can be updated when new observations become available and refined by additional criteria to display the outcome ranges of more specific subgroups. Modal values and medians provide further reference points for five types of outcome assessment: strongly pessimistic, weakly pessimistic, average, weakly optimistic, and strongly optimistic.

This graphical representation makes variation within subgroups visible instead of reducing each subgroup to a single representative outcome. Its interpretation nevertheless depends on the observations and on the reference interval chosen for the nomogram. Observed extrema do not bound all possible future outcomes, and the numerical components of an intuitionistic fuzzy pair change if the reference interval changes. Successive refinement also reduces the number of observations in each branch. Subgroup sizes should therefore accompany the displayed ranges and pairs, particularly when comparing narrowly defined subgroups. The pessimistic and optimistic assessments additionally require an explicit definition of which direction of the outcome is favourable.

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